Student's Manual

Essential Mathematics for Economic Analysis

3rd edition

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Preface

This student's solutions manual accompanies *Essential Mathematics for Economic Analysis* (3rd edition, FT Prentice Hall, 2008). Its main purpose is to provide more detailed solutions to the problems marked (SM) in the text. This Manual should be used in conjunction with the answers in the text. In some few cases only a part of the problem is done in detail, because the rest follows the same pattern. We are grateful to Carren Pindiriri for help with the proofreading. We would appreciate suggestions for improvements from our readers as well as help in weeding out inaccuracies and errors.

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Contents

Chapter 1 Introductory Topics I: Algebra

1.1

1. (a) True (b) False. -5 is smaller than -3, so on the number line it is to the left of -3. (See Fig. 1.1.1 in the book.) (c) False. -13 is an integer, but not a natural number. (d) True. Any natural number is rational. For example $5 = 5/1$. (e) False, since $3.1415 = 31415/10000$, the quotient of two integers. (f) False. Counterexample: $\sqrt{2} + (-\sqrt{2}) = 0$. (g) True. (h) True.

1.3

9. (a) $(2t-1)(t^2-2t+1) = 2t(t^2-2t+1)-(t^2-2t+1) = 2t^3-4t^2+2t-t^2+2t-1 = 2t^3-5t^2+4t-1$
(b) $(a+1)^2 + (a-1)^2 - 2(a+1)(a-1) = a^2 + 2a + 1 + a^2 - 2a + 1 - 2a^2 + 2 = 4$
(c) $(x+y+z)^2 = (x+y+z)(x+y+z) = x(x+y+z) + y(x+y+z) + z(x+y+z) = x^2 + xy + xz + yx + y^2 + yz + zx + zy + z^2 = x^2 + y^2 + z^2 + 2xy + 2xz + 2yz$ (d) $(x-y-z)^2 = (x-y-z)(x-y-z) = x^2-xy-xz-xy+y^2-yz-xz-yz+z^2$, so $(x+y+z)^2-(x-y-z)^2 = 4xy+4xz$

13. (a) $a^2+4ab+4b^2 = (a+2b)^2$ using the first quadratic identity. (d) $9z^2-16w^2 = (3z-4w)(3z+4w)$, according to the difference-of-squares formula. (e) $-\frac{1}{5}x^2 + 2xy - 5y^2 = -\frac{1}{5}(x^2 - 10xy + 25y^2) = -\frac{1}{5}(x-5y)^2$ (f) $a^4 - b^4 = (a^2-b^2)(a^2+b^2)$, using the difference-of-squares formula. Since $a^2 - b^2 = (a-b)(a+b)$, the answer in the book follows.

1.4

5. (a) $\dfrac{1}{x-2} - \dfrac{1}{x+2} = \dfrac{x+2}{(x-2)(x+2)} - \dfrac{x-2}{(x+2)(x-2)} = \dfrac{x+2-x+2}{(x-2)(x+2)} = \dfrac{4}{x^2-4}$

(b) Since $4x+2 = 2(2x+1)$ and $4x^2-1 = (2x+1)(2x-1)$, the LCD is $2(2x+1)(2x-1)$. Then
$\dfrac{6x+25}{4x+2} - \dfrac{6x^2+x-2}{4x^2-1} = \dfrac{(6x+25)(2x-1) - 2(6x^2+x-2)}{2(2x+1)(2x-1)} = \dfrac{21(2x-1)}{2(2x+1)(2x-1)} = \dfrac{21}{2(2x+1)}$

(c) $\dfrac{18b^2}{a^2-9b^2} - \dfrac{a}{a+3b} + 2 = \dfrac{18b^2 - a(a-3b) + 2(a^2-9b^2)}{(a+3b)(a-3b)} = \dfrac{a(a+3b)}{(a+3b)(a-3b)} = \dfrac{a}{a-3b}$

(d) $\dfrac{1}{8ab} - \dfrac{1}{8b(a+2)} + \dfrac{1}{b(a^2-4)} = \dfrac{a^2-4-a(a-2)+8a}{8ab(a^2-4)} = \dfrac{2(5a-2)}{8ab(a^2-4)} = \dfrac{5a-2}{4ab(a^2-4)}$

(e) $\dfrac{2t-t^2}{t+2} \cdot \left(\dfrac{5t}{t-2} - \dfrac{2t}{t-2}\right) = \dfrac{t(2-t)}{t+2} \cdot \dfrac{3t}{t-2} = \dfrac{-t(t-2)}{t+2} \cdot \dfrac{3t}{t-2} = \dfrac{-3t^2}{t+2}$

(f) $\dfrac{a\left(1-\frac{1}{2a}\right)}{0.25} = \dfrac{a-\frac{1}{2}}{\frac{1}{4}} = 4a-2$, so $2 - \dfrac{a\left(1-\frac{1}{2a}\right)}{0.25} = 2-(4a-2) = 4-4a = 4(1-a)$

6. (a) $\dfrac{2}{x} + \dfrac{1}{x+1} - 3 = \dfrac{2(x+1)+x-3x(x+1)}{x(x+1)} = \dfrac{2-3x^2}{x(x+1)}$

(b) $\dfrac{t}{2t+1} - \dfrac{t}{2t-1} = \dfrac{t(2t-1)-t(2t+1)}{(2t+1)(2t-1)} = \dfrac{-2t}{4t^2-1}$

(c) $\dfrac{3x}{x+2} - \dfrac{4x}{2-x} - \dfrac{2x-1}{(x-2)(x+2)} = \dfrac{3x(x-2)+4x(x+2)-(2x-1)}{(x-2)(x+2)} = \dfrac{7x^2+1}{x^2-4}$

(d) $\dfrac{\dfrac{1}{x}+\dfrac{1}{y}}{\dfrac{1}{xy}} = \dfrac{\left(\dfrac{1}{x}+\dfrac{1}{y}\right)xy}{\dfrac{1}{xy}\cdot xy} = \dfrac{y+x}{1} = x+y$ (e) $\dfrac{\dfrac{1}{x^2}-\dfrac{1}{y^2}}{\dfrac{1}{x^2}+\dfrac{1}{y^2}} = \dfrac{\left(\dfrac{1}{x^2}-\dfrac{1}{y^2}\right)\cdot x^2y^2}{\left(\dfrac{1}{x^2}+\dfrac{1}{y^2}\right)\cdot x^2y^2} = \dfrac{y^2-x^2}{x^2+y^2}$

(f) Multiply the numerator and the denominator by xy. Then the fraction reduces to $\dfrac{a(y-x)}{a(y+x)} = \dfrac{y-x}{y+x}$.

8. (a) $\frac{1}{4}-\frac{1}{5}=\frac{5}{20}-\frac{4}{20}=\frac{1}{20}$, so $\left(\frac{1}{4}-\frac{1}{5}\right)^{-2}=\left(\frac{1}{20}\right)^{-2}=20^2=400$

(b) $n-\dfrac{n}{1-\dfrac{1}{n}} = n-\dfrac{n\cdot n}{\left(1-\dfrac{1}{n}\right)\cdot n} = n-\dfrac{n^2}{n-1} = \dfrac{n(n-1)-n^2}{n-1} = \dfrac{-n}{n-1}$

(c) If $u=x^{p-q}$, then $\dfrac{1}{1+x^{p-q}}+\dfrac{1}{1+x^{q-p}} = \dfrac{1}{1+u}+\dfrac{1}{1+1/u} = \dfrac{1}{1+u}+\dfrac{u}{1+u} = 1$

(d) $\dfrac{\left(\dfrac{1}{x-1}+\dfrac{1}{x^2-1}\right)(x^2-1)}{\left(x-\dfrac{2}{x+1}\right)(x^2-1)} = \dfrac{x+1+1}{x^3-x-2x+2} = \dfrac{x+2}{(x+2)(x^2-2x+1)} = \dfrac{1}{(x-1)^2}$

(e) $\dfrac{1}{(x+h)^2}-\dfrac{1}{x^2} = \dfrac{x^2-(x+h)^2}{x^2(x+h)^2} = \dfrac{-2xh-h^2}{x^2(x+h)^2}$, so $\dfrac{\dfrac{1}{(x+h)^2}-\dfrac{1}{x^2}}{h} = \dfrac{-2x-h}{x^2(x+h)^2}$

(f) Multiplying denominator and numerator by $x^2-1=(x+1)(x-1)$ yields $\dfrac{10x^2}{5x(x-1)} = \dfrac{2x}{x-1}$

1.5

5. (Needs some hints.) Multiply the denominator and the numerator by: (a) $\sqrt{7}-\sqrt{5}$ (b) $\sqrt{5}-\sqrt{3}$ (c) $\sqrt{3}+2$ (d) $x\sqrt{y}-y\sqrt{x}$ (e) $\sqrt{x+h}+\sqrt{x}$ (f) $1-\sqrt{x+1}$

7. The answers will depend on the calculator you use.

12. (a) For $x=1$ the left-hand side is 4 and the right-hand side is 2. (In fact, $(2^x)^2=2^{2x}$.) (b) Correct because $a^{p-q}=a^p/a^q$ (c) Correct because $a^{-p}=1/a^p$ (d) For $x=1$ it says $5=1/5$, which is absurd. (e) For $x=y=1$, it says that $a^2=2a$, which is usually wrong. (In fact, $a^{x+y}=a^xa^y$.) (f) $2^{\sqrt{x}}\cdot 2^{\sqrt{y}}=2^{\sqrt{x}+\sqrt{y}}$, not $2^{\sqrt{xy}}$.

1.6

4. (a) $2<\dfrac{3x+1}{2x+4}$ has the same solutions as $\dfrac{3x+1}{2x+4}-2>0$, or $\dfrac{3x+1-2(2x+4)}{2x+4}>0$, or $\dfrac{-x-7}{2x+4}>0$ A sign diagram reveals that the inequality is satisfied for $-7<x<-2$. A serious error is to multiply the inequality by $2x+4$, without assuming that $2x+4>0$. When multiplying with $2x+4$ when this number is negative, the inequality sign must be reversed. (It might be a good idea to test the inequality for some values of x. For example, for $x=0$ it is not true. What about $x=-5$?)

(b) The inequality is equivalent to $\dfrac{120}{n}-\dfrac{3}{4}\le 0$, i.e. $\dfrac{3(160-n)}{4n}\le 0$. A sign diagram reveals that the inequality is satisfied for $n<0$ and for $n\ge 160$. (Note that for $n=0$ the inequality makes no sense. For $n=160$, we have equality.)

(c) Easy: $g(g-2) \leq 0$ etc. (d) Note that $p^2 - 4p + 4 = (p-2)^2$, and the inequality reduces to $\dfrac{p+1}{(p-2)^2} \geq 0$. The fraction makes no sense if $p = 2$. The conclusion follows.

(e) The inequality is equivalent to $\dfrac{-n-2}{n+4} - 2 \geq 0$, i.e. $\dfrac{-n-2-2n-8}{n+4} \geq 0$, or $\dfrac{-3n-10}{n+4} \geq 0$, etc.

(f) See the text and use a sign diagram. (Don't cancel x^2. If you do, $x = 0$ appears as a false solution.)

5. (a) Use a sign diagram. (b) The inequality is not satisfied for $x = 1$. If $x \neq 1$, it is obviously satisfied for $x + 4 > 0$, i.e. $x > -4$ (because $(x-1)^2$ is positive when $x \neq 1$). (c) Use a sign diagram. (d) The inequality is not satisfied for $x = 1/5$. If $x \neq 1/5$, it is obviously satisfied for $x < 1$.

(e) Use a sign diagram. ($(5x-1)^{11} < 0$ if $x < 1/5$, > 0 if $x > 1/5$.)

(f) $\dfrac{3x-1}{x} > x + 3$, $\dfrac{3x-1}{x} - (x+3) > 0$, $\dfrac{-(1+x^2)}{x} > 0$, so $x < 0$. ($1 + x^2$ is always positive.)

(g) $\dfrac{x-3}{x+3} > 2x - 1$ or $\dfrac{x-3}{x+3} - (2x-1) < 0$ or $\dfrac{-2x(x+2)}{x+3} < 0$. Then use a sign diagram.

(h) Use the hint and a sign diagram. (Actually, this problem and the next could be postponed to Section 2.3 if you have forgotten your high school algebra.) (i) Use the hint and a sign diagram.

Review Problems for Chapter 1

4. (a) $(2x)^4 = 2^4x^4 = 16x^4$ (b) $2^{-1} - 4^{-1} = 1/2 - 1/4 = 1/4$, so $(2^{-1} - 4^{-1})^{-1} = 4$

(c) Cancel the common factor $4x^2yz^2$. (d) $-(-ab^3)^{-3} = -(-1)^{-3}a^{-3}b^{-9} = a^{-3}b^{-9}$, so $[-(-ab^3)^{-3}(a^6b^6)^2]^3 = [a^{-3}b^{-9}a^{12}b^{12}]^3 = [a^9b^3]^3 = a^{27}b^9$ (e) $\dfrac{a^5 \cdot a^3 \cdot a^{-2}}{a^{-3} \cdot a^6} = \dfrac{a^6}{a^3} = a^3$

(f) $\left[\left(\dfrac{x}{2}\right)^3 \cdot \dfrac{8}{x^{-2}}\right]^{-3} = \left[\dfrac{x^3 \cdot 8}{8 \cdot x^{-2}}\right]^{-3} = (x^5)^{-3} = x^{-15}$

8. All are straightforward, except (c), (g), and (h): (c) $-\sqrt{3}\left(\sqrt{3} - \sqrt{6}\right) = -3 + \sqrt{3}\sqrt{6} = -3 + \sqrt{3}\sqrt{3}\sqrt{2}$ $= -3 + 3\sqrt{2}$ (g) $(1 + x + x^2 + x^3)(1 - x) = (1 + x + x^2 + x^3) - (1 + x + x^2 + x^3)x = 1 - x^4$

(h) $(1+x)^4 = (1+x)^2(1+x)^2 = (1 + 2x + x^2)(1 + 2x + x^2)$ a.s.o.

11. (a) and (b) are easy. (c) $ax + ay + 2x + 2y = ax + 2x + ay + 2y = (a+2)x + (a+2)y = (a+2)(x+y)$

(d) $2x^2 - 5yz + 10xz - xy = 2x^2 + 10xz - (xy + 5yz) = 2x(x + 5z) - y(x + 5z) = (2x - y)(x + 5z)$

(e) $p^2 - q^2 + p - q = (p-q)(p+q) + (p-q) = (p-q)(p+q+1)$ (f) See the answer in the book.

15. (a) $\dfrac{s}{2s-1} - \dfrac{s}{2s+1} = \dfrac{s(2s+1) - s(2s-1)}{(2s-1)(2s+1)} = \dfrac{2s}{4s^2-1}$

(b) $\dfrac{x}{3-x} - \dfrac{1-x}{x+3} - \dfrac{24}{x^2-9} = \dfrac{-x(x+3) - (1-x)(x-3) - 24}{(x-3)(x+3)} = \dfrac{-7(x+3)}{(x-3)(x+3)} = \dfrac{-7}{x-3}$

(c) Multiplying numerator and denominator by x^2y^2 yields, $\dfrac{y-x}{y^2-x^2} = \dfrac{y-x}{(y-x)(y+x)} = \dfrac{1}{x+y}$

16. (a) Cancel the factor $25ab$. (b) $x^2 - y^2 = (x+y)(x-y)$. Cancel $x + y$. (c) The fraction can be written $\dfrac{(2a-3b)^2}{(2a-3b)(2a+3b)} = \dfrac{2a-3b}{2a+3b}$. (d) $\dfrac{4x - x^3}{4 - 4x + x^2} = \dfrac{x(2-x)(2+x)}{(2-x)^2} = \dfrac{x(2+x)}{2-x}$

Chapter 2 Introductory Topics II: Equations

2.1

3. (a) We note first that $x = -3$ and $x = -4$ both make the equation absurd. Multiplying the equation by the common denominator, $(x+3)(x+4)$, yields $(x-3)(x+4) = (x+3)(x-4)$, and thus $x = 0$. (b) Multiplying by the common denominator $(x-3)(x+3)$ yields $3(x+3) - 2(x-3) = 9$, from which we get $x = -6$. (c) Multiplying by the common denominator $15x$ (assuming that $x \neq 0$, yields $18x^2 - 75 = 10x^2 - 15x + 8x^2$, from which we get $x = 5$.

5. (a) Multiplying by the common denominator 12 yields $9y - 3 - 4 + 4y + 24 = 36y$, and so $y = 17/23$. (b) Multiplying by $2x(x+2)$ yields $8(x+2) + 6x = 2(2x+2) + 7x$, from which we find $x = -4$. (c) Multiplying the numerator and the denominator in the first fraction by $1-z$, leads to $\dfrac{2-2z-z}{(1-z)(1+z)} = \dfrac{6}{2z+1}$. Multiplying by $(1-z^2)(2z+1)$ yields $(2-3z)(2z+1) = 6 - 6z^2$, and so $z = 4$.
(d) Expanding the parentheses we get $\frac{p}{4} - \frac{3}{8} - \frac{1}{4} + \frac{p}{12} - \frac{1}{3} + \frac{p}{3} = -\frac{1}{3}$. Multiplying by the common denominator 24 gives an equation with the solution $p = 15/16$.

2.2

2. (a) Multiply both sides by abx to obtain $b+a = 2abx$. Hence, $x = \dfrac{a+b}{2ab} = \dfrac{a}{2ab} + \dfrac{b}{2ab} = \dfrac{1}{2}\left(\dfrac{1}{a} + \dfrac{1}{b}\right)$.
(b) Multiply the equation by $cx + d$ to obtain $ax + b = cAx + dA$, or $(a - cA)x = dA - b$, and thus $x = (dA - b)/(a - cA)$. (c) Multiply the equation by $x^{1/2}$ to obtain $\frac{1}{2}p = wx^{1/2}$, thus $x^{1/2} = p/2w$, so, by squaring each side, $x = p^2/4w^2$. (d) Multiply each side by $\sqrt{1+x}$ to obtain $1 + x + ax = 0$, so $x = -1/(1+a)$. (e) $x^2 = b^2/a^2$, so $x = \pm b/a$. (f) We see immediately that $x = 0$.

4. (a) $\alpha x - a = \beta x - b \iff (\alpha - \beta)x = a - b$, so $x = (a-b)/(\alpha - \beta)$.
(b) Squaring each side of $\sqrt{pq} = 3q + 5$ yields $pq = (3q+5)^2$, so $p = (3q+5)^2/q$.
(c) $Y = 94 + 0.2(Y - (20 + 0.5Y)) = 94 + 0.2Y - 4 - 0.1Y$, so $0.9Y = 90$, and then $Y = 100$.
(d) Raise each side to the 4th power: $K^2 \frac{1}{2} \dfrac{r}{w} K = Q^4$, so $K^3 = 2wQ^4/r$, and hence $K = \left(2wQ^4/r\right)^{1/3}$.
(e) Multiplying numerator and denominator in the left-hand fraction by $4K^{1/2}L^{3/4}$, leads to $2L/K = r/w$, from which we get $L = rK/2w$. (f) Raise each side to the 4th power: $\frac{1}{16}p^4K^{-1}\left(\frac{1}{2}\dfrac{r}{w}\right) = r^4$. It follows that $K^{-1} = 32r^3w/p^4$, so $K = \frac{1}{32}p^4r^{-3}w^{-1}$.

5. (a) $\dfrac{1}{s} = \dfrac{1}{t} - \dfrac{1}{T} = \dfrac{T-t}{tT}$, so $s = \dfrac{tT}{T-t}$. (b) $\sqrt{KLM} = B + \alpha L$, so $KLM = (B + \alpha L)^2$, and so $M = (B + \alpha L)^2/KL$. (c) Multiplying each side by $x - z$ yields, $x - 2y + xz = 4xy - 4yz$, or $(x+4y)z = 4xy - x + 2y$, and so $z = (4xy - x + 2y)/(x + 4y)$. (d) $V = C - CT/N$, so $CT/N = C - V$ and thus $T = N(1 - V/C)$.

2.3

5. (a) See the answer in the book. (b) If the first natural number is n, then the next is $n+1$, so the requirement is that $n^2 + (n+1)^2 = 13$, which reduces to $2n^2 + 2n - 12 = 0$, i.e. $n^2 + n - 6 = 0$. This second-order equation has the solutions $n = -3$ and $n = 2$, so the two numbers are 2 and 3. (If we asked for integer solutions, we would have -3 and -2 in addition.)

(c) If the shortest side is x, the other is $x + 14$, so according to Pythagoras' Theorem (see page 633 and draw a picture), $x^2 + (x + 14)^2 = (34)^2$, or $x^2 + 14x - 480 = 0$. The solutions are $x = 16$ and $x = -30$, so the shortest side is 16 cm and the longest is 30 cm. (d) If the usual driving speed is x km/h and the usual time spent is t hours, then $xt = 80$. 16 minutes is $16/60 = 4/15$ hours, so driving at the speed $x + 10$ for $t - 4/15$ hours gives $(x + 10)(t - 4/15) = 80$. From the first equation, $t = 80/x$. Inserting this into the second equation, we get $(x + 10)(80/x - 4/15) = 80$. Rearranging, we obtain $x^2 + 10x - 3000 = 0$, whose positive solution is $x = 50$. So his usual driving speed is 50 km/h.

2.4

4. (a) If the two numbers are x and y, then $x + y = 52$ and $x - y = 26$. Adding the two equations gives $2x = 78$, so $x = 39$, and then $y = 52 - 39 = 13$. (b) Let the cost of one chair be \$$x$ and the cost of one table \$$y$. Then $5x + 20y = 1800$ and $2x + 3y = 420$. Solving this system yields $x = 120$, $y = 60$.
(c) Units produced of B: x. Then $x + \frac{1}{2}x = \frac{3}{2}x$ units are produced of A, and $300 \cdot \frac{3}{2}x + 200x = 13\,000$, or $650x = 13\,000$, so $x = 20$. Thus, 30 of quality A and 20 of quality B should be produced.
(d) If she invests \$$x$ at 15 % and \$$y$ at 20 %, then $x + y = 1500$ and $0.15x + 0.2y = 275$. The solution is $x = 8000$ and $y = 2000$.

2.5

2. (a) The numerator $5 + x^2$ is never 0, so there are no solutions. (b) The equation is obviously equivalent to $\dfrac{x^2 + 1 + 2x}{x^2 + 1} = 0$, or $\dfrac{(x + 1)^2}{x^2 + 1} = 0$, so $x = -1$. (c) $x = -1$ is clearly no solution. Multiply the equation by $(x + 1)^{2/3}$. Then the denominator becomes $x + 1 - \frac{1}{3}x$, which is 0 for $x = -3/2$.
(d) Multiplying by $x - 1$ and rearranging yields $x(2x - 1) = 0$, and so $x = 0$ or $x = 1/2$.

3. (a) $z = 0$ satisfies the equation. If $z \neq 0$, canceling z^2 yields $z - a = za + zb$, or $z(1 - (a + b)) = a$. If $a + b = 1$ we have a contradiction. If $a + b \neq 1$, $z = a/(1 - (a + b))$. (b) The equation is equivalent to $(1 + \lambda)\mu(x - y) = 0$, so $\lambda = -1$, $\mu = 0$, or $x = y$. (c) $\mu = \pm 1$ makes the equation meaningless. Multiplying the equation by $1 - \mu^2$ yields $\lambda(1 - \mu) = -\lambda$, or $\lambda(2 - \mu) = 0$, so $\lambda = 0$ or $\mu = 2$.
(d) The equation is equivalent to $b(1 + \lambda)(a - 2) = 0$, so $b = 0$, $\lambda = -1$, or $a = 2$.

Review Problems for Chapter 2

2. See Problem 2.1.3.

3. (a) $x = \frac{2}{3}(y - 3) + y = \frac{2}{3}y - 2 + y = \frac{5}{3}y - 2$, or $\frac{5}{3}y = x + 2$, so $y = \frac{3}{5}(x + 2)$.
(b) $ax - cx = b + d$, or $(a - c)x = b + d$, so $x = (b + d)/(a - c)$.
(c) $\sqrt{L} = Y_0/AK$, so squaring each side yields $L = (Y_0/AK)^2$. (d) $qy = m - px$, so $y = (m - px)/q$.
(e) and (f): See the answers in the text.

5. (a) Multiply the equation by $5K^{1/2}$ to obtain $K^{1/2} = 15L^{1/3}$. Squaring each side gives $K = 225L^{2/3}$.
(b) Raise each side to the power $1/t$ to obtain $1 + r/100 = 2^{1/t}$, and so $r = 100(2^{1/t} - 1)$.
(c) $abx_0^{b-1} = p$, so $x_0^{b-1} = p/ab$. Now raise each side to the power $1/(b - 1)$.
(d) Raise each side to the power $-\rho$ to get $(1 - \lambda)a^{-\rho} + \lambda b^{-\rho} = c^{-\rho}$, or $b^{-\rho} = \lambda^{-1}(c^{-\rho} - (1 - \lambda)a^{-\rho})$. Now raise each side to the power $-1/\rho$.

9. (a) See the answer in the text. (b) Let $u = 1/x$ and $v = 1/y$. Then the system reduces to $3u + 2v = 2$, $2u - 3v = 1/4$, with solution $u = 1/2$, $v = 1/4$. It follows that $x = 1/u = 2$ and $y = 1/v = 4$.
(c) See the answer in the text.

Chapter 3 Introductory Topics II: Miscellaneous

3.1

3. (a)–(d): Look at the last term and replace n by k. Sum over k from 1 to n. (e) The coefficients are the powers 3^n for $n = 1, 2, 3, 4, 5$, so the general term is $3^n x^n$. (f) and (g) see answers in the text.
(h) This is tricky. One has to see that each term is 198 larger that the previous term. (The problem is related to the story about Gauss on page 56.)

7. (a) $\sum_{k=1}^{n} ck^2 = c \cdot 1^2 + c \cdot 2^2 + \cdots + c \cdot n^2 = c(1^2 + 2^2 + \cdots + n^2) = c\sum_{k=1}^{n} k^2$
(b) Wrong even for $n = 2$: The left-hand side is $(a_1 + a_2)^2 = a_1^2 + 2a_1a_2 + a_2^2$, but the right-hand side is $a_1^2 + a_2^2$. (c) Both sides equal $b_1 + b_2 + \cdots + b_N$. (d) Both sides equal $5^1 + 5^2 + 5^3 + 5^4 + 5^5$.
(e) Both sides equal $a_{0,j}^2 + \cdots + a_{n-1,j}^2$. (f) Wrong even for $n = 2$: The left-hand side is $a_1 + a_2/2$, but the right-hand side is $(1/k)(a_1 + a_2)$.

3.2

5. One does not have to use summation signs. The sum is $a + (a + d) + (a + 2d) + \cdots + (a + (n - 1)d$. There are n terms. The sum of all the a's is na. The rest is $d(1 + 2 + \cdots + n - 1)$. Then use formula (4).

3.3

1. (a) See the text. (b) $\sum_{s=0}^{2}\sum_{r=2}^{4}\left(\frac{rs}{r+s}\right)^2 = \sum_{s=0}^{2}\left[\left(\frac{2s}{2+s}\right)^2 + \left(\frac{3s}{3+s}\right)^2 + \left(\frac{4s}{4+s}\right)^2\right] = \left(\frac{2}{3}\right)^2 + \left(\frac{3}{4}\right)^2 + \left(\frac{4}{5}\right)^2 + \left(\frac{4}{4}\right)^2 + \left(\frac{6}{5}\right)^2 + \left(\frac{8}{6}\right)^2 = 5 + \frac{3113}{3600}$
(c) $\sum_{i=1}^{m}\sum_{j=1}^{n} i \cdot j^2 = \sum_{i=1}^{m} i \cdot \sum_{j=1}^{n} j^2 = \frac{1}{2}m(m+1) \cdot \frac{1}{6}n(n+1)(2n+1) = \frac{1}{12}m(m+1)n(n+1)(2n+1)$, where we used (4) and (5).

4. $\bar{a}$ is the mean of the $\bar{a}_s$'s because $\bar{a} = \dfrac{1}{n}\displaystyle\sum_{s=1}^{n}\left(\frac{1}{m}\sum_{r=1}^{m} a_{rs}\right) = \frac{1}{n}\sum_{s=1}^{n}\bar{a}_s$.

To prove $(*)$, note that because $a_{rj} - \bar{a}$ is independent of the summation index s, it is a common factor when we sum over s, so $\sum_{s=1}^{m}(a_{rj} - \bar{a})(a_{sj} - \bar{a}) = (a_{rj} - \bar{a})\sum_{s=1}^{m}(a_{sj} - \bar{a})$ for each r. Next, summing over r gives

$$\sum_{r=1}^{m}\sum_{s=1}^{m}(a_{rj} - \bar{a})(a_{sj} - \bar{a}) = \left[\sum_{r=1}^{m}(a_{rj} - \bar{a})\right]\left[\sum_{s=1}^{m}(a_{sj} - \bar{a})\right] \qquad (**)$$

Using the properties of sums and the definition of $\bar{a}_j$, we have

$$\sum_{r=1}^{m}(a_{rj} - \bar{a}) = \sum_{r=1}^{m} a_{rj} - \sum_{r=1}^{m}\bar{a} = m\bar{a}_j - m\bar{a} = m(\bar{a}_j - \bar{a})$$

Similarly, replacing r with s as the index of summation, one also has $\sum_{s=1}^{m}(a_{sj} - \bar{a}) = m(\bar{a}_j - \bar{a})$. Substituting these values into $(**)$ then confirms $(*)$.

3.4

6. (a) If (i) $\sqrt{x - 4} = \sqrt{x + 5} - 9$, then also (ii) $x - 4 = (\sqrt{x + 5} - 9)^2$, which we get by squaring both sides in (i). Calculating the square on the right-hand side of (ii) gives $\sqrt{x + 5} = 5$, and so $x + 5 = 25$,

i.e. $x = 20$. This shows that *if* x is a solution of (i), then $x = 20$. No other value of x can satisfy (i). *But if we check this solution*, we find that with $x = 20$ the LHS of (i) becomes $\sqrt{16} = 4$, and the RHS becomes $\sqrt{25} - 9 = 5 - 9 = -4$. Thus the LHS and the RHS are different. This means that equation (i) actually has no solutions at all. (But note that $4^2 = (-4)^2$, i.e. the square of the LHS equals the square of the RHS. That is how the spurious solution $x = 20$ managed to sneak in.)
(b) If x is a solution of (iii) $\sqrt{x-4} = 9 - \sqrt{x+5}$, then just as in part (a) we find that x must be a solution of (iv) $x - 4 = (9 - \sqrt{x+5}\,)^2$. Now, $(9 - \sqrt{x+5}\,)^2 = (\sqrt{x+5} - 9)^2$, so equation (iv) is equivalent to equation (ii) in part (a). This means that (iv) has exactly one solution, namely $x = 20$. Inserting this value of x into equation (iii), we find that $x = 20$ *is* a solution of (iii).
A geometric explanation of the results can be given with reference to the following figure.

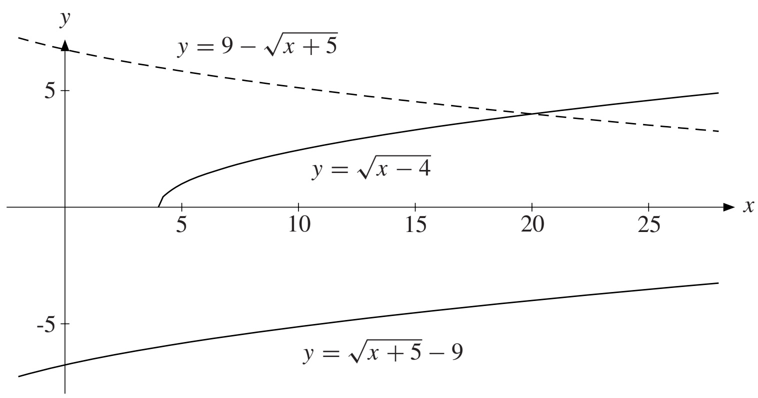


Figure SM3.4.6

We see that the two solid curves in the figure have no point in common, that is, the expressions $\sqrt{x-4}$ and $\sqrt{x+5} - 9$ are not equal for any value of x. (In fact, the difference $\sqrt{x-4} - (\sqrt{x+5} - 9)$ increases with x, so there is no point of intersection farther to the right, either.) This explains why the equation in (a) has no solution. The dashed curve $y = 9 - \sqrt{x+5}$, on the other hand, intersects $y = \sqrt{x+5}$ for $x = 20$ (and only there), and this corresponds to the solution in part (b).
Comment: In part (a) it was necessary to check the result, because the transition from (i) to (ii) is only an implication, not an equivalence. Similarly, it was necessary to check the result in part (b), since the transition from (iii) to (iv) also is only an implication — at least, it is not clear that it is an equivalence. (Afterwards, it turned out to be an equivalence, but we could not know that until we had solved the equation.)

7. (a) Here we have "iff" since $\sqrt{4} = 2$. (b) It is easy to see by means of a sign diagram that $x(x+3) < 0$ precisely when x lies in the open interval $(-3, 0)$. Therefore we have an implication from left to right (that is, "only if"), but not in the other direction. (For example, if $x = 10$, then $x(x+3) = 130$.) (c) $x^2 < 9 \iff -3 < x < 3$, so $x^2 < 9$ only if $x < 3$. If $x = -5$, for instance, we have $x < 3$ but $x^2 > 9$. Hence we cannot have "if" here. (d) $x^2 + 1$ is never 0, so we have "iff" here. (e) If $x > 0$, then $x^2 > 0$, but $x^2 > 0$ also when $x < 0$. (f) $x^4 + y^4 = 0 \iff x = 0$ and $y = 0$. If $x = 0$ and, say, $y = 1$, then $x^4 + y^4 = 1$, so we cannot have "if" here.

9. (a) If x and y are not both nonnegative, at leat one of them must be negative, i.e. $x < 0$ or $y < 0$.
(b) If not all x are greater than or equal to a, at least one x must be less than a. (c) At least one of them is less than 5. (Would it be easier if the statement to negate were "Neither John nor Diana is less than 5 years old"?) (d)–(f) See the answers in the text.

3.7

3. For $n = 1$, both sides are 1/2. Suppose (∗) is true for $n = k$. Then the sum of the first $k + 1$ terms is

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}$$

But $\dfrac{k}{k+1}+\dfrac{1}{(k+1)(k+2)}=\dfrac{(k+1)^2}{(k+1)(k+2)}=\dfrac{k+1}{k+2}$, which is (∗) for $n = k+1$. Thus, by induction, (∗) is true for all n.

4. The claim is true for $n = 1$. As the induction hypothesis, suppose $k^3 + (k + 1)^3 + (k + 2)^3$ is divisible by 9. Note that $(k + 1)^3 + (k + 2)^3 + (k + 3)^3 = (k + 1)^3 + (k + 2)^3 + k^3 + 9k^2 + 27k + 27 = k^3 + (k+1)^3 + (k+2)^3 + 9(k^2 + 3k + 3)$. This is divisible by 9 because the induction hypothesis implies that the sum of the first three terms is divisible by 9, whereas the last term is also obviously divisible by 9.

Review Problems for Chapter 3

6. (b) ⇒ false (because $x^2 = 16$ also has the solution $x = -4$), ⇐ true, because if $x = 4$, then $x^2 = 16$. (c) ⇒ true, ⇐ false because with $y > -2$ and $x = 3$, $(x - 3)^2(y + 2) = 0$. (d) ⇒ and ⇐ both true, since the equation $x^3 = 8$ has the solution $x = 2$ and no others. (In the terminology of Section 6.3, $f(x) = x^3$ is strictly increasing. See Problem 6.3.3 and see the graph Fig. 7, page 88.)

9. Consider Fig. A3.6.8, page 643 in the book, and let n_k denote the number of students in the set marked S_k, for $k = 1, 2, \dots, 8$. Sets A, B, and C refer to those who study English, French, and Spanish, respectively. Since 10 students take all three languages, $n_7 = 10$. There are 15 who take French and Spanish, so $15 = n_2 + n_7$, and thus $n_2 = 5$. Furthermore, $32 = n_3 + n_7$, so $n_3 = 22$. Also, $110 = n_1 + n_7$, so $n_1 = 100$. The rest of the information implies that $52 = n_2+n_3+n_6+n_7$, so $n_6 = 52-5-22-10 = 15$. Moreover, $220 = n_1+n_2+n_5+n_7$, so $n_5 = 220-100-5-10 = 105$. Finally, $780 = n_1+n_3+n_4+n_7$, so $n_4 = 780 - 100 - 22 - 10 = 648$. The answers to the problems are:
(a): $n_1 = 100$ (b): $n_3 + n_4 = 648 + 22 = 670$ (c) $1000 - \sum_{i=1}^{8} n_i = 1000 - 905 = 95$

Chapter 4 Functions of One Variable

4.2

1. (a) $f(0) = 0^2 + 1 = 1$, $f(-1) = (-1)^2 + 1 = 2$, $f(1/2) = (1/2)^2 + 1 = 1/4 + 1 = 5/4$, and $f(\sqrt{2}) = (\sqrt{2})^2 + 1 = 2 + 1 = 3$. (b) (i) Since $(-x)^2 = x^2$, $f(x) = f(-x)$ for all x. (ii) $f(x+1) = (x+1)^2+1 = x^2+2x+1+1 = x^2+2x+2$ and $f(x)+f(1) = x^2+1+2 = x^2+3$. Thus equality holds if and only if $x^2+2x+2 = x^2+3$, i.e. if and only if $x = 1/2$. (iii) $f(2x) = (2x)^2+1 = 4x^2 + 1$ and $2f(x) = 2x^2 + 2$. Now, $4x^2 + 1 = 2x^2 + 2 \Leftrightarrow x^2 = 1/2 \Leftrightarrow x = \pm\sqrt{1/2} = \pm\frac{1}{2}\sqrt{2}$.

10. (a) No: $f(2+1) = f(3) = 18$, whereas $f(2)+f(1) = 10$. (b) Yes: $f(2+1) = f(2)+f(1) = -9$. (c) No: $f(2 + 1) = f(3) = \sqrt{3} \approx 1.73$, whereas $f(2) + f(1) = \sqrt{2} + 1 \approx 2.41$.

13. (a) We must require $5 - x \geq 0$, so $x \leq 5$. (b) The denominator $x^2 - x = x(x - 1)$ must be different from 0, so $x \neq 0$ and $x \neq 1$. (c) To begin with, the denominator must be nonzero, so we must require $x \neq 2$ and $x \neq -3$. Moreover, since we can only take the square root of a nonnegative number, the fraction $(x - 1)/(x - 2)(x + 3)$ must be ≥ 0. A sign diagram reveals that $D_f = (-3, 1] \cup (2, \infty)$. Note in particular that the function is defined with value 0 at $x = 1$.

15. Since g is obviously defined for $x \geq -2$, $D_g = [-2, \infty)$. Note that $g(-2) = 1$, and $g(x) \leq 1$ for all $x \in D_f$. As x increases from -2 to ∞, $g(x)$ decreases from 1 to $-\infty$, so $R_g = (-\infty, 1]$.

4.4

3. If $D = a + bP$, then $200 = a + 10b$, and $150 = a + 15b$. Solving for a and b yields $a = 300$ and $b = -10$, so $D = 300 - 10P$.

4. L_1: The slope is obviously 1, and the point-slope formula with $(x_1, y_1) = (0, 2)$ and $a = 1$ give $y = x+2$. L_2: Using the point-point formula with $(x_1, y_1) = (0, 3)$ and $(x_2, y_2) = (5, 0)$ yields: $y - 3 = \dfrac{0-3}{5-0}x$, or $y = -\frac{3}{5}x + 3$. L_3: Has slope 0 and equation $y = 1$. For L_4 and L_5 see the text.

10. The set of points that satisfy the inequality $3x + 4y \leq 12$ are those on or below the straight line $3x + 4y = 12$ as explained in Example 6 for a similar inequality. Those points that satisfy the inequality $x - y \leq 1$, or equivalently, $y \geq x - 1$, are those on or above the straight line $x - y = 1$. Finally, those points that satisfy the inequality $3x + y \geq 3$, or equivalently, $y \geq 3 - 3x$, are those on or above the straight line $3x + y = 3$. The set of points that satisfy all these three inequalities simultaneously, is the set shown in Fig. A.4.4.10.

4.5

3. The point–point formula gives $C - 200 = \dfrac{275 - 200}{150 - 100}(x - 100)$, or $C = \dfrac{3}{2}x + 50$.

4.6

2. Complementing the answers in the text. (c) Formula (4) with $a = -\frac{1}{2}$ and $b = -1$ gives $x = -1$ as the maximum point. (Alternatively, completing the square, $f(x) = -\frac{1}{2}(x^2+2x-3) = -\frac{1}{2}(x^2+2x+1-4) = -\frac{1}{2}(x+1)^2+2$, from which we see immediately that $f(x)$ has maximum 2 at $x = -1$.) (e) Use (2.3.5), or expand, to verify the formula for $f(x)$. Use a sign diagram to study the sign variation of $f(x)$.

6. Expanding we get $U(x) = -(1 + r^2)x^2 + 8(r - 1)x$. Then apply (4.6.4) with $a = -(1 + r^2)$ and $b = 8(r - 1)$.

9. (b) If $B^2 - 4AC > 0$, then according to formula (2.3.4), the equation $f(x) = Ax^2 + Bx + C = 0$ would have two distinct solutions, which is impossible when $f(x) \geq 0$ for all x. We find that $A = a_1^2 + a_2^2 + \cdots + a_n^2$, $B = 2(a_1b_1 + a_2b_2 + \cdots + a_nb_n)$, and $C = b_1^2 + b_2^2 + \cdots + b_n^2$, so the conclusion follows easily.

4.7

1. (a) Integer roots must divide 6. Thus ±1, ±2, ±3, and ±6 are the only possible integer solutions. We find that $-2, -1, 1, 3$ all are roots, and since there can be no more than 4 roots in a polynomial equation of degree 4, we have found them all.
(b) The same possible integer solutions. Only -6 and 1 are integer solutions. (The third root is $-1/2$.)
(c) Neither 1 nor -1 satisfies the equation, so there are no integer roots.
(d) First multiply the equation by 4 to have integer coefficients. Then ±1, ±2, and ±4 are seen to be the only possible integer solutions. In fact, 1, 2, -2 are all solutions.

3. (a) The answer is $2x^2 + 2x + 4 + 3/(x-1)$, because

$$
\begin{array}{rl}
(2x^3 \phantom{{}-2x^2} + 2x - 1) \div (x-1) = 2x^2 + 2x + 4 & \\
\underline{2x^3 - 2x^2 \phantom{{}+2x-1}} & \\
2x^2 + 2x - 1 & \\
\underline{2x^2 - 2x \phantom{{}-1}} & \\
4x - 1 & \\
\underline{4x - 4} & \\
3 & \text{remainder}
\end{array}
$$

(b) The answer is $x^2 + 1$, because

$$
\begin{array}{rl}
(x^4 + x^3 + x^2 + x) \div (x^2 + x) = x^2 + 1 & \\
\underline{x^4 + x^3 \phantom{{}+x^2+x}} & \\
x^2 + x & \\
\underline{x^2 + x} & \\
0 & \text{no remainder}
\end{array}
$$

(c) The answer is $x^3 - 4x^2 + 3x + 1 - 4x/(x^2 + x + 1)$, because

$$
\begin{array}{rl}
(x^5 - 3x^4 \phantom{{}-x^3-4x^2+3x} + 1) \div (x^2 + x + 1) = x^3 - 4x^2 + 3x + 1 & \\
\underline{x^5 + x^4 + x^3 \phantom{{}+x^2+x+1}} & \\
-4x^4 - x^3 \phantom{{}+x^2+x} + 1 & \\
\underline{-4x^4 - 4x^3 - 4x^2 \phantom{{}+x+1}} & \\
3x^3 + 4x^2 \phantom{{}+3x} + 1 & \\
\underline{3x^3 + 3x^2 + 3x \phantom{{}+1}} & \\
x^2 - 3x + 1 & \\
\underline{x^2 + x + 1} & \\
-4x \phantom{{}+1} & \text{remainder}
\end{array}
$$

(d) The answer is $3x^5 + 6x^3 - 3x^2 + 12x - 12 + (28x^2 - 36x + 13)/(x^3 - 2x + 1)$, because

$$
\begin{array}{rl}
(3x^8 \phantom{{}-6x^6+3x^5-12x^4+6x^3} + x^2 \phantom{{}-12x} + 1) \div (x^3 - 2x + 1) = 3x^5 + 6x^3 - 3x^2 + 12x - 12 & \\
\underline{3x^8 - 6x^6 + 3x^5 \phantom{{}-12x^4+6x^3+x^2-12x+1}} & \\
6x^6 - 3x^5 \phantom{{}-12x^4+6x^3} + x^2 \phantom{{}-12x} + 1 & \\
\underline{6x^6 \phantom{{}-3x^5} - 12x^4 + 6x^3 \phantom{{}+x^2-12x+1}} & \\
-3x^5 + 12x^4 - 6x^3 + x^2 \phantom{{}-12x} + 1 & \\
\underline{-3x^5 \phantom{{}+12x^4} + 6x^3 - 3x^2 \phantom{{}-12x+1}} & \\
12x^4 - 12x^3 + 4x^2 \phantom{{}-12x} + 1 & \\
\underline{12x^4 \phantom{{}-12x^3} - 24x^2 + 12x \phantom{{}+1}} & \\
-12x^3 + 28x^2 - 12x + 1 & \\
\underline{-12x^3 \phantom{{}+28x^2} + 24x - 12} & \\
28x^2 - 36x + 13 & \text{remainder}
\end{array}
$$

4. (a) $y = \frac{1}{2}(x+1)(x-3)$. (Since the graph intersects the x-axis at the two points $x = -1$ and $x = 3$, we try the quadratic function $f(x) = a(x+1)(x-3)$. Then $f(1) = -4a$, and since the graph passes through the point $(1, -2)$,. $f(1) = -2 = -4a$. But the $a = 1/2$.) (b) Because the equation $f(x) = 0$ has roots $x = -3, 1, 2$, we try the cubic function $f(x) = b(x+3)(x-1)(x-2)$. Then $f(0) = 6b$. According to the graph, $f(0) = -12$. So $b = -2$, and hence $y = -2(x+3)(x-1)(x-2)$. (c) $y = \frac{1}{2}(x+3)(x-2)^2$. (We try a polynomial of the form $f(x) = c(x-2)^2(x+3)$, with $x = 2$ as a double root. Then $f(0) = 12c$. From the graph we see that $f(0) = 6$, and so $a = 1/2$.)

8. Polynomial division gives

$$
\begin{array}{llll}
(x^2 & -\gamma x & &) \div (x+\beta) = x - (\beta+\gamma) \\
x^2 & +\beta x & & \\
\hline
 & -(\beta+\gamma)x & & \\
 & -(\beta+\gamma)x - \beta(\beta+\gamma) & & \\
\hline
 & \qquad\qquad \beta(\beta+\gamma) & \text{remainder} &
\end{array}
$$

and so

$$E = \alpha\Big(x - (\beta+\gamma) + \frac{\beta(\beta+\gamma)}{x+\gamma}\Big) = \alpha x - \alpha(\beta+\gamma) + \frac{\alpha\beta(\beta+\gamma)}{x+\gamma}$$

4.8

4. (a) C. The graph is a parabola and since the coefficient in front of x^2 is positive, it has a minimum point.
(b) D. The function is defined for $x \le 2$ and crosses the y-axis at $y = 2\sqrt{2} \approx 2.8$.
(c) E. The graph is a parabola and since the coefficient in front of x^2 is negative, it has a maximum point.
(d) B. When x increases, y decreases, and y becomes close to -2 when x is large.
(e) A. The function is defined for $x \ge 2$ and increases as x increases.
(f) F. Let $y = 2 - (\frac{1}{2})^x$. Then y increases as x increases. For large values of x, y is close to 2.

5. (a) See the answer in the text. (b) $9^t = (3^2)^t = 3^{2t}$ and $(27)^{1/5}/3 = (3^3)^{1/5}/3 = 3^{3/5}/3 = 3^{-2/5}$, and then $2t = -2/5$, so $t = -1/5$.

4.9

10. Suppose $y = Ab^x$, with $b > 0$. Then in (a), since the graph passes through the points $(x, y) = (0, 2)$ and $(x, y) = (2, 8)$, we get $2 = Ab^0$, or $A = 2$, and $8 = 2b^2$, so $b = 2$. Hence, $y = 2 \cdot 2^x$.
In (b), $\frac{2}{3} = Ab^{-1}$ and $6 = Ab$. It follows that $A = 2$ and $b = 3$, and so $y = 2 \cdot 3^x$.
In (c), $4 = Ab^0$ and $1/4 = Ab^4$. It follows that $A = 4$ and $b^4 = 1/16$, and so $b = 1/2$. Thus, $y = 4(\frac{1}{2})^x$.

4.10

3. (a) and (c) see the text. (b) Since $\ln x^2 = 2\ln x$, $7\ln x = 6$, so $\ln x = 6/7$, and thus $x = e^{6/7}$.

4. (a) $\ln(Ae^{rt}) = \ln(Be^{st})$, so $\ln A + rt = \ln B + st$, or $(r-s)t = \ln(B/A)$, and so $t = \dfrac{1}{r-s}\ln\dfrac{B}{A}$.
(b) $t = \dfrac{1}{0.09 - 0.02}\ln\dfrac{5.6 \cdot 10^{12}}{1.2 \cdot 10^{12}} = \dfrac{1}{0.07}\ln\dfrac{14}{3} \approx 22.$
According to this, the two countries would have the same GNP in approximately 22 years, so in 2012.

Review Problems for Chapter 4

4. (a) We must have $x^2 \geq 1$, i.e. $x \geq 1$ or $x \leq -1$. (Look at Fig. 4.3.6, page 88.)
(b) The square root is defined if $x \geq 4$, but $x = 4$ makes the denominator 0, so we must require $x > 4$.
(c) We must have $(x-3)(5-x) \geq 0$, i.e. $3 \leq x \leq 5$ (using a sign diagram).

7. (a) The point-slope formula gives $y - 3 = -3(x+2)$, or $y = -3x - 3$.
(b) The two-point formula gives: $y - 5 = \dfrac{7-5}{2-(-3)}(x-(-3))$, or $y = 2x/5 + 31/5$.
(c) $y - b = \dfrac{3b-b}{2a-a}(x-a)$, or $y = (2b/a)x - b$.

10. $(1,-3)$ belongs to the graph if $-3 = a + b + c$, $(0,-6)$ belongs to the graph if $-6 = c$, and $(3, 15)$ belongs to the graph if $15 = 9a + 3b + c$. It follows that $a = 2$, $b = 1$, and $c = -6$.

14. (a) $p(x) = x(x^2 + x - 12) = x(x-3)(x+4)$, because $x^2 + x - 12 = 0$ for $x = 3$ and $x = -4$.
(b) $\pm1, \pm2, \pm4, \pm8$ are the only possible integer zeros. By trial and error we find that $q(2) = q(-4) = 0$, so $2(x-2)(x+4) = 2x^2 + 4x - 16$ is a factor for $q(x)$. By polynomial division we find that $q(x) \div (2x^2 + 4x - 16) = x - 1/2$, so $q(x) = 2(x-2)(x+4)(x-1/2)$.

16. We use (4.7.5) and denote each polynomial by $p(x)$. (a) $p(2) = 8 - 2k = 0$ for $k = 4$.
(b) $p(-2) = 4k^2 + 2k - 6 = 0$ for $k = -3/2$ and $k = 1$. (c) $p(-2) = -26 + k = 0$ for $k = 26$.
(d) $p(1) = k^2 - 3k - 4 = 0$ for $k = -1$ and $k = 4$.

17. Since $p(2) = 0$, $x - 2$ is a factor in $p(x)$. We find that $p(x) \div (x-2) = \frac{1}{4}(x^2 - 2x - 15) = \frac{1}{4}(x+3)(x-5)$, so $x = -3$ and $x = 5$ are the two other zeros. (Alternative: $q(x)$ has the same zeros as $4p(x) = x^3 - 4x^2 - 11x + 30$. This polynomial can only have $\pm1, \pm2, \pm3, \pm5, \pm10, \pm15$, and ±30 as integer zeros. It is tedious work to find the zeros in this way.)

21. (a) $\ln(x/e^2) = \ln x - \ln e^2 = \ln x - 2$ (b) $\ln(xz/y) = \ln(xz) - \ln y = \ln x + \ln z - \ln y$
(c) $\ln(e^3x^2) = \ln e^3 + \ln x^2 = 3 + 2\ln x$ for $x > 0$. (In general, $\ln x^2 = 2\ln|x|$.) (d) See the text.

Chapter 5 Properties of Functions

5.1

3. The equilibrium condition is $106 - P = 10 + 2P$, and thus $P = 32$. The corresponding quantity is $Q = 106 - 32 = 74$. See the graph in the answer section of the text.

6. $f(y^* - d) = f(y^*) - c$ gives $A(y^* - d) + B(y^* - d)^2 = Ay^* + B(y^*)^2 - c$, or $Ay^* - Ad + B(y^*)^2 - 2Bdy^* + Bd^2 = Ay^* + B(y^*)^2 - c$. It follows that $y^* = [Bd^2 - Ad + c]/2Bd$.

5.2

4. If $f(x) = 3x + 7$, then $f(f(x)) = f(3x+7) = 3(3x+7) + 7 = 9x + 28$. $f(f(x^*)) = 100$ requires $9x^* + 28 = 100$, and so $x^* = 8$.

5.3

4. (a) f does have an inverse since it is one-to-one. This is shown in the table by the fact that all numbers in the second row, the domain of f^{-1}, are different. The inverse assigns to each number in the second row, the corresponding number in the first row. (b) Since $f(0) = 4$ and $f(x)$ increases by 2 for each unit increase in x, $f(x) = 2x + 4$. Solving $y = 2x + 4$ for x yields $x = \frac{1}{2}y - 2$, and thus $f^{-1}(x) = \frac{1}{2}x - 2$.

9. (a) $(x^3-1)^{1/3}=y \iff x^3-1=y^3 \iff x=(y^3+1)^{1/3}$. If we use x as the independent variable, $f^{-1}(x)=(x^3+1)^{1/3}$. $\mathbb{R}$ is the domain and range for both f and f^{-1}.

(b) The domain of f is all $x \neq 2$ Hence (with $x \neq 2$), $\dfrac{x+1}{x-2}=y \iff x+1=y(x-2) \iff (1-y)x=-2y-1 \iff x=\dfrac{-2y-1}{1-y}=\dfrac{2y+1}{y-1}$. Using x as the independent variable, $f^{-1}(x)=(2x+1)/(x-1)$. The domain of the inverse is all $x \neq 1$.

(c) Here $y=(1-x^3)^{1/5}+2 \iff y-2=(1-x^3)^{1/5} \iff (y-2)^5=1-x^3 \iff x^3=1-(y-2)^5 \iff x=(1-(y-2)^5)^{1/3}$. With x as the free variable, $f^{-1}(x)=(1-(x-2)^5)^{1/3}$. $\mathbb{R}$ is the domain and range for both f and f^{-1}.

10. (a) The domain is $\mathbb{R}$ and the range is $(0,\infty)$, so the inverse is defined on $(0,\infty)$. From $y=e^{x+4}$, $\ln y=x+4$, so $x=\ln y-4$, $y>0$. (b) The range is $\mathbb{R}$, which is the domain of the inverse. From $y=\ln x-4$, $\ln x=y+4$, and then $x=e^{y+4}$. (c) The domain is $\mathbb{R}$. y is increasing and when $x\to-\infty$, $y\to\ln 2$. Moreover, $y\to\infty$ as $x\to\infty$. So the range of the function is $(\ln 2,\infty)$. From $y=\ln\bigl(2+e^{x-3}\bigr)$, $2+e^{x-3}=e^y$, so $e^{x-3}=e^y-3$, and thus $x=3+\ln(e^y-3)$, $y>\ln 2$.

11. We must solve $x=\frac{1}{2}(e^y-e^{-y})$ for y. Multiply the equation by e^y to get $\frac{1}{2}e^{2y}-\frac{1}{2}=xe^y$ or $e^{2y}-2xe^y-1=0$. Letting $e^y=z$ yields $z^2-2xz-1=0$, with solution $z=x\pm\sqrt{x^2+1}$. The minus sign makes z negative, so $z=e^y=x+\sqrt{x^2+1}$. This gives $y=\ln\bigl(x+\sqrt{x^2+1}\bigr)$ as the inverse function.

5.4

1. (a) It is natural first to see if the curve intersects the axes, by putting $x=0$, and then $y=0$. This gives 4 points. Then choose some values of $-\sqrt{6}<x<\sqrt{6}$, and compute the corresponding values of y. Argue why the graph is symmetric about the x-axis and the y-axis. (The curve is called an ellipse. See the next section.) (b) This graph is also symmetric about the x-axis and the y-axis. (If (a,b) lies on the graph, so does $(a,-b)$, $(-a,b)$, and $(-a,-b)$. (The graph is a hyperbola. See the next section.)

2. We see that we must have $x\geq 0$ and $y\geq 0$. If (a,b) lies on the graph, so does (b,a), so the graph is symmetric about the line $y=x$. See the answer in the text.

5.5

4. (a) See the text. (b) Since the circle has centre at (2, 5), its equation is $(x-2)^2+(x-5)^2=r^2$. Since $(-1,3)$ lies on the circle, $(-1-2)^2+(3-5)^2=r^2$, so $r^2=13$.

8. $x^2+y^2+Ax+By+C=0 \iff x^2+Ax+y^2+By+C=0 \iff x^2+Ax+\bigl(\frac{1}{2}A\bigr)^2+y^2+By+\bigl(\frac{1}{2}B\bigr)^2=\frac{1}{4}(A^2+B^2-4C) \iff \bigl(x+\frac{1}{2}A\bigr)^2+\bigl(y+\frac{1}{2}B\bigr)^2=\frac{1}{4}(A^2+B^2-4C)$. The last is the equation of a circle centred at $\bigl(-\frac{1}{2}A,-\frac{1}{2}B\bigr)$ with radius $\frac{1}{2}\sqrt{A^2+B^2-4C}$. If $A^2+B^2=4C$, the graph consists only of the point $\bigl(-\frac{1}{2}A,-\frac{1}{2}B\bigr)$. For $A^2+B^2<4C$, the solution set is empty.

5.6

1. In each case, except (c), the rule defines a function because it associates with each member of the original set a unique member in the target set. For instance, in (d), if the surface area of a sphere is given, its volume is uniquely determined: From $S=4\pi r^2$, $r=(S/4\pi)^{1/2}$, and then $V=\frac{4}{3}\pi r^3=\frac{4}{3}\pi(S/4\pi)^{3/2}$ (for the formulas for the surface area and the volume of a sphere of radius r, see page 632.)

Review Problems for Chapter 5

3. (a) Equilibrium condition: $150 - \frac{1}{2}P^* = 20 + 2P^*$, which gives $P^* = 52$ and $Q^* = 20 + 2P^* = 124$. For (b) and (c) se answers in the text.

7. (a) f is defined and strictly increasing for $e^x > 2$, i.e. $x > \ln 2$. Its range is $\mathbb{R}$. ($f(x) \to -\infty$ as $x \to \ln 2^+$, and $f(x) \to \infty$ as $x \to \infty$.) From $y = 3 + \ln(e^x - 2)$, we get $\ln(e^x - 2) = y - 3$, and so $e^x - 2 = e^{y-3}$, or $e^x = 2 + e^{y-3}$, so $x = \ln(2 + e^{y-3})$. Hence $f^{-1}(x) = \ln(2 + e^{x-3})$, $x \in \mathbb{R}$.
(b) Note that f is strictly increasing. Moreover, $e^{-\lambda x} \to \infty$ as $x \to -\infty$, and $e^{-\lambda x} \to 0$ as $x \to \infty$. Therefore, $f(x) \to 0$ as $x \to -\infty$, and $f(x) \to 1$ as $x \to \infty$. So the range of f, and therefore the domain of f^{-1}, is $(0, 1)$. From $y = \dfrac{a}{e^{-\lambda x} + a}$ we get $e^{-\lambda x} + a = a/y$, so $e^{-\lambda x} = a(1/y - 1)$, or $-\lambda x = \ln a + \ln(1/y - 1)$. Thus $x = -(1/\lambda)\ln a - (1/\lambda)\ln(1/y - 1)$, and therefore the inverse is $f^{-1}(x) = -(1/\lambda)\ln a - (1/\lambda)\ln(1/x - 1)$, with $x \in (0, 1)$.

Chapter 6 Differentiation

6.2

5. (a) We start by using the recipe in (6.2.3) to find the slope of the tangent.
(a) (A): $f(a + h) = f(0 + h) = 3h + 2$ (B): $f(a + h) - f(a) = f(h) - f(0) = 3h + 2 - 2 = 3h$ (C)–(D): $[f(h) - f(0)]/h = 3$ (E): $[f(h) - f(0)]/h = 3 \to 3$ as $h \to 0$, so $f'(0) = 3$. The slope of the tangent at $(0, 2)$ is 3.
(b) (A): $f(a+h) = f(1+h) = (1+h)^2 - 1 = 1+2h+h^2 - 1 = 2h+h^2$ (B): $f(1+h) - f(1) = 2h+h^2$ (C)–(D): $[f(1+h) - f(1)]/h = 2+h$ (E): $[f(1+h) - f(1)]/h = 2+h \to 2$ as $h \to 0$, so $f'(1) = 2$.
(c) (A): $f(3+h) = 2+3/(3+h)$ (B): $f(3+h) - f(3) = 2+3/(3+h) - 3 = -h/(3+h)$ (C)–(D): $[f(3+h) - f(3)]/h = -1/(3+h)$ (E): $[f(3+h) - f(3)]/h = -1/(3+h) \to -1/3$ as $h \to 0$, so $f'(3) = -1/3$. (d) $[f(h) - f(0)]/h = (h^3 - 2h)/h = h^2 - 2 \to -2$ as $h \to 0$, so $f'(0) = -2$.
(e) $\dfrac{f(-1+h) - f(-1)}{h} = -\dfrac{1 + h + 1/(-1+h) + 2}{h} = \dfrac{h}{-1+h} \to 0$ as $h \to 0$, so $f'(0) = 0$.
(f) $\dfrac{f(1+h) - f(1)}{h} = \dfrac{(1+h)^4 - 1}{h} = \dfrac{h^4 + 4h^3 + 6h^2 + 4h + 1 - 1}{h} = h^3 + 4h^2 + 6h + 4 \to 4$ as $h \to 0$, so $f'(1) = 4$.

8. (a) $\big(\sqrt{x+h} - \sqrt{x}\big)\big(\sqrt{x+h} + \sqrt{x}\big) = (\sqrt{x+h})^2 + \sqrt{x+h}\,\sqrt{x} - \sqrt{x}\sqrt{x+h} - (\sqrt{x})^2 = x + h - x = h$.
(b) $\dfrac{f(x+h) - f(x)}{h} = \dfrac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})} = \dfrac{h}{h(\sqrt{x+h} + \sqrt{x})} = \dfrac{1}{\sqrt{x+h} + \sqrt{x}}$
(c) This follows from (b).

6.5

5. (a) $\dfrac{1/3 - 2/3h}{h-2} = \dfrac{3h\big(1/3 - 2/3h\big)}{3h(h-2)} = \dfrac{h-2}{3h(h-2)} = \dfrac{1}{3h} \to \dfrac{1}{6}$ as $h \to 2$
(b) When $x \to 0$, $x^2 - 1 \to -1$ and $x^2 \to \infty$, so the fraction has no limit, but tends to $-\infty$.
(c) $\dfrac{32t - 96}{t^2 - 2t - 3} = \dfrac{32(t-3)}{(t-3)(t+1)} = \dfrac{32}{t+1} \to 8$, as $t \to 3$, so $\sqrt[3]{\dfrac{32t - 96}{t^2 - 2t - 3}} \to \sqrt[3]{8} = 2$ as $t \to 3$.
(d) $\dfrac{\sqrt{h+3} - \sqrt{3}}{h} = \dfrac{(\sqrt{h+3} - \sqrt{3})(\sqrt{h+3} + \sqrt{3})}{h(\sqrt{h+3} + \sqrt{3})} = \dfrac{h + 3 - 3}{h(\sqrt{h+3} + \sqrt{3})} = \dfrac{1}{\sqrt{h+3} + \sqrt{3}} \to \dfrac{1}{2\sqrt{3}}$ as $h \to 0$.

(e) $\dfrac{t^2-4}{t^2+10t+16} = \dfrac{(t+2)(t-2)}{(t+2)(t+8)} = \dfrac{t-2}{t+8} \to -\dfrac{2}{3}$ as $t \to -2$.

(f) Observe that $4-x = (2+\sqrt{x})(2-\sqrt{x})$, so $\lim\limits_{x\to 4} \dfrac{2-\sqrt{x}}{4-x} = \lim\limits_{x\to 4} \dfrac{1}{2+\sqrt{x}} = \dfrac{1}{4}$.

6. (a) $\dfrac{f(x)-f(1)}{x-1} = \dfrac{x^2+2x-3}{x-1} = \dfrac{(x-1)(x+3)}{x-1} = x+3 \to 4$ as $x \to 1$.

(b) $\dfrac{f(x)-f(1)}{x-1} = x+3 \to 5$ as $x \to 2$.

(c) $\dfrac{f(2+h)-f(2)}{h} = \dfrac{(2+h)^2+2(2+h)-8}{h} = \dfrac{h^2+6h}{h} = h+6 \to 6$ as $h \to 0$.

(d) $\dfrac{f(a+h)-f(a)}{h} = \dfrac{(a+h)^2+2(a+h)-a^2-2a}{h} = 2a+2+h \to 2a+2$ as $h \to 0$.

(e) Same answer as in (d) putting $x-a=h$.

(f) $\dfrac{f(a+h)-f(a-h)}{h} = \dfrac{(a+h)^2+2a+2h-(a-h)^2-2a+2h}{h} = 4a+4 \to 4a+4$ as $h \to 0$.

7. (a) $x^3-8=0$ has the solution $x=2$, and polynomial division yields $x^3-8=(x-2)(x^2+2x+4)$. (b) and (c): see the text.

6.6

7. (a) With $f(x)=x^2$, $\lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)}{h} = \lim\limits_{h\to 0} \dfrac{(5+h)^2-5^2}{h} = f'(5)$. On the other hand, $f'(x) = 2x$, so $f'(5)=10$, and the limit is therefore 10. (b) and (c): see the text.

6.7

3. (a) $y = \dfrac{1}{x^6} = x^{-6} \Rightarrow y' = -6x^{-7}$, using the power rule (6.6.4).

(b) $y = x^{-1}(x^2+1)\sqrt{x} = x^{-1}x^2x^{1/2} + x^{-1}x^{1/2} = x^{3/2}+x^{-1/2} \Rightarrow y' = \frac{3}{2}x^{1/2} - \frac{1}{2}x^{-3/2}$

(c) $y = x^{-3/2} \Rightarrow y' = -\frac{3}{2}x^{-5/2}$ (d) $y = \dfrac{x+1}{x-1} \Rightarrow y' = \dfrac{1\cdot(x-1)-(x+1)\cdot 1}{(x-1)^2} = \dfrac{-2}{(x-1)^2}$

(e) $y = \dfrac{x}{x^5} + \dfrac{1}{x^5} = x^{-4}+x^{-5} \Rightarrow y' = -\dfrac{4}{x^5} - \dfrac{5}{x^6}$

(f) $y = \dfrac{3x-5}{2x+8} \Rightarrow \dfrac{3(2x+8)-2(3x-5)}{(2x+8)^2} = \dfrac{34}{(2x+8)^2}$ (g) $y = 3x^{-11} \Rightarrow y' = -33x^{-12}$

(h) $y = \dfrac{3x-1}{x^2+x+1} \Rightarrow y' = \dfrac{3(x^2+x+1)-(3x-1)(2x+1)}{(x^2+x+1)^2} = \dfrac{-3x^2+2x+4}{(x^2+x+1)^2}$

6. (a) $f'(x) = 6x-12 = 6(x-2) \ge 0 \iff x \ge 2$, so f is increasing in $[2,\infty)$. (b) $f'(x) = x^3-3x = x(x^2-3) = x(x-\sqrt{3})(x+\sqrt{3})$, so (using a sign diagram) f is increasing in $\big[-\sqrt{3},0\big]$ and in $\big[\sqrt{3},\infty\big)$.

(c) $f'(x) = \dfrac{2(2-x^2)}{(x^2+2)^2} = \dfrac{2(2-\sqrt{2})(2+\sqrt{2})}{(x^2+2)^2}$, so f is increasing in $[-\sqrt{2},\sqrt{2}]$. (d) See the text.

7. (a) $y' = -1-2x = -3$ when $x=1$, so the slope of the tangent is -3. Since $y=1$ when $x=1$, the point-slope formula gives $y-1 = -3(x-1)$, or $y = -3x+4$. (b) $y' = 4x/(x^2+1)^2 = 1$ and $y=0$ when $x=1$, so $y = x-1$. (c) $y = x^2-x^{-2}$, so $y' = 2x+2x^{-3} = 17/4$ and $y = 15/4$ when $x=2$, so $y = (17/4)x - 19/4$. (d) $y' = \dfrac{4x^3(x^3+3x^2+x+3)-(x^4+1)(3x^2+6x+1)}{[(x^2+1)(x+3)]^2} = -\dfrac{1}{19}$ and $y = 1/3$ when $x=0$, so $y = -(x-3)/9$.

9. (a) We use the quotient rule: $y = \dfrac{at+b}{ct+d} \;\Rightarrow\; y' = \dfrac{a(ct+d)-(at+b)c}{(ct+d)^2} = \dfrac{ad-bc}{(ct+d)^2}$

(b) $y = t^n\left(a\sqrt{t}+b\right) = at^{n+1/2} + bt^n \;\Rightarrow\; y' = (n+1/2)at^{n-1/2} + nbt^{n-1}$.

(c) $y = \dfrac{1}{at^2+bt+c} \;\Rightarrow\; y' = \dfrac{0\cdot(at^2+bt+c) - 1\cdot(2at+b)}{(at^2+bt+c)^2} = \dfrac{-2at-b}{(at^2+bt+c)^2}$

12. This is rather tricky because the denominator is 0 at $x_{1,2} = 2 \pm \sqrt{2}$. A sign diagram shows that $f(x) > 0$ only in $(-\infty, 0)$ and in (x_1, x_2).

6.8

3. (a) $y = (x^2+x+1)^{-5} = u^{-5}$, where $u = x^2+x+1$. By the chain rule, $y' = (-5)u^{-6}u' = -5(2x+1)(x^2+x+1)^{-6}$. (b) With $u = x + \sqrt{x+\sqrt{x}}$, $y = \sqrt{u} = u^{1/2}$, so $y' = \frac{1}{2}u^{-1/2}u'$. Now, $u = x + v^{1/2}$, with $v = x + x^{1/2}$. Then $u' = 1 + \frac{1}{2}v^{-1/2}v'$, where $v' = 1 + \frac{1}{2}x^{-1/2}$. Thus, all in all, $y' = \frac{1}{2}u^{-1/2}u' = \frac{1}{2}\big[x + (x+x^{1/2})^{1/2}\big]^{-1/2}(1 + \frac{1}{2}(x+x^{1/2})^{-1/2}(1+\frac{1}{2}x^{-1/2}))$. (c) See the text.

6. $x = b - \sqrt{ap-c} = b - \sqrt{u}$, with $u = ap - c$. Then $\dfrac{dx}{dp} = -\dfrac{1}{2\sqrt{u}}u' = -\dfrac{a}{2\sqrt{ap-c}}$.

12. (a), (e), and (g) are easy. For the others, you need the chain rule. In (d) you need the differentiation rules for sum, product as well as the chain rule. See the text.

6.9

4. $g'(t) = \dfrac{2t(t-1)-t^2}{(t-1)^2} = \dfrac{t^2-2t}{(t-1)^2}$, $g''(t) = \dfrac{(2t-2)(t-1)^2 - (t^2-2t)2(t-1)}{(t-1)^4} = \dfrac{2(t-1)}{(t-1)^4} = \dfrac{2}{(t-1)^3}$, so $g''(2) = 2$.

5. With simplified notation: $y' = f'g + fg'$, $y'' = f''g + f'g' + f'g' + fg'' = f''g + 2f'g' + fg''$, $y''' = f'''g + f''g' + 2f''g' + 2f'g'' + f'g'' + fg''' = f'''g + 3f''g' + 3f'g'' + fg'''$

6.10

2. (a) $dx/dt = (b+2ct)e^t + (a+bt+ct^2)e^t = (a+b+(b+2c)t+ct^2)e^t$

(b) $\dfrac{dx}{dt} = \dfrac{3qt^2te^t - (p+qt^3)(e^t+te^t)}{t^2e^{2t}} = \dfrac{(-qt^4+2qt^3-pt-p)e^t}{t^2e^{2t}}$ (c) See the text.

4. (a) $y' = 3x^2 + 2e^{2x}$ is obviously positive everywhere, so y increases in $(-\infty, \infty)$.

(b) $y' = 10xe^{-4x} + 5x^2(-4)e^{-4x} = 10x(1-2x)e^{-4x}$. A sign diagram shows that y increases in $[0, 1/2]$.

(c) $y' = 2xe^{-x^2} + x^2(-2x)e^{-x^2} = 2x(1-x)(1+x)e^{-x^2}$. A sign diagram shows that y increases in $(-\infty, -1]$ and in $[0, 1]$. (The answer in the text is wrong.)

6.11

3. For these problems we need the chain rule. That is an important rule! In particular, we need the fact that $\dfrac{d}{dx}\ln f(x) = \dfrac{1}{f(x)}f'(x) = \dfrac{f'(x)}{f(x)}$ when f is a differentiable function (with $f(x) > 0$).

(a) $y = \ln(\ln x) = \ln u \;\Longrightarrow\; y' = \dfrac{1}{u}u' = \dfrac{1}{\ln x}\dfrac{1}{x} = \dfrac{1}{x\ln x}$.

(b) $y = \ln\sqrt{1-x^2} = \ln u \;\Longrightarrow\; y' = \dfrac{1}{u}u' = \dfrac{1}{\sqrt{1-x^2}}\dfrac{-2x}{2\sqrt{1-x^2}} = \dfrac{-x}{1-x^2}$.

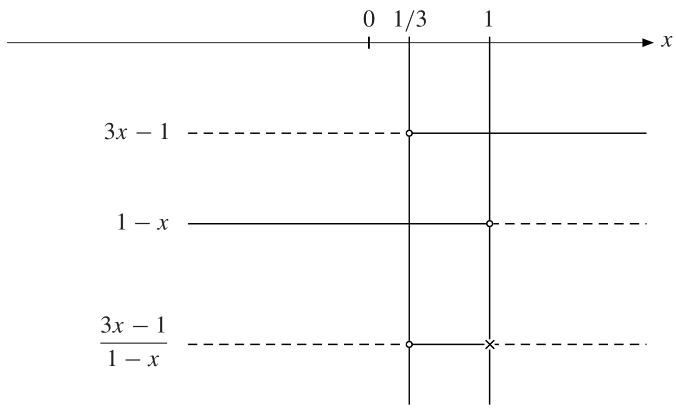
(Alternatively: $\sqrt{1-x^2} = (1-x^2)^{1/2} \;\Longrightarrow\; y = \frac{1}{2}\ln(1-x^2)$, and so on.)

(c) $y = e^x \ln x \implies y' = e^x \ln x + e^x \dfrac{1}{x} = e^x\Big(\ln x + \dfrac{1}{x}\Big)$.

(d) $y = e^{x^3} \ln x^2 \implies y' = 3x^2 e^{x^3} \ln x^2 + e^{x^3}\dfrac{1}{x^2}2x = e^{x^3}\Big(3x^2 \ln x^2 + \dfrac{2}{x}\Big)$.

(e) $y = \ln(e^x + 1) \implies y' = \dfrac{e^x}{e^x + 1}$. (f) $y = \ln(x^2 + 3x - 1) \implies y' = \dfrac{2x + 3}{x^2 + 3x - 1}$.

4. (a) $\ln u$ is defined for $u > 0$, so we must require $x + 1 > 0$, i.e. $x > 0$.
(b) We must have $1 - x \neq 0$ for the fraction to be defined, and $\dfrac{3x - 1}{1 - x} > 0$ for the logarithm to be defined. A sign diagram (see below) shows that $\dfrac{3x - 1}{1 - x}$ is defined and positive if and only if $1/3 < x < 1$.
(c) $\ln |x|$ is defined $\iff |x| > 0 \iff x \neq 0$.



5. (a) One must have $x^2 > 1$, i.e. $x > 1$ or $x < -1$. (See Fig. 4.3.6 in the text.) (b) $\ln(\ln x)$ is defined when $\ln x$ is defined and positive, that is, for $x > 1$. (c) The fraction $\dfrac{1}{\ln(\ln x) - 1}$ is defined when $\ln(\ln x)$ is defined and different from 1. From (b), $\ln(\ln x)$ is defined when $x > 1$. Further, $\ln(\ln x) = 1 \iff \ln x = e \iff x = e^e$. Conclusion: $\dfrac{1}{\ln(\ln x) - 1}$ is defined $\iff x > 1$ and $x \neq e^e$.

9. In these problems we can use logarithmic differentiation. Alternatively we can write the functions in the form $f(x) = e^{g(x)}$ and then use the fact that $f'(x) = e^{g(x)}g'(x) = f(x)g'(x)$.
(a) Let $f(x) = (2x)^x$. Then $\ln f(x) = x \ln(2x)$, so $\dfrac{f'(x)}{f(x)} = 1 \cdot \ln(2x) + x \cdot \dfrac{1}{2x} \cdot 2 = \ln(2x) + 1$. Hence, $f'(x) = f(x)(\ln(2x) + 1) = (2x)^x(\ln x + \ln 2 + 1)$.
(b) $f(x) = x^{\sqrt{x}} = \big(e^{\ln x}\big)^{\sqrt{x}} = e^{\sqrt{x}\ln x}$, so $f'(x) = e^{\sqrt{x}\ln x} \cdot \dfrac{d}{dx}(\sqrt{x}\ln x) = x^{\sqrt{x}}\left(\dfrac{\ln x}{2\sqrt{x}} + \dfrac{\sqrt{x}}{x}\right)$
(c) $\ln f(x) = x \ln \sqrt{x} = \frac{1}{2}x \ln x$, so $f'(x)/f(x) = \frac{1}{2}(\ln x + 1)$, which gives $f'(x) = \frac{1}{2}(\sqrt{x}\,)^x(\ln x + 1)$.

10. $\ln y = v \ln u$, so $y'/y = v' \ln u + (v/u)u'$, and so $y' = u^v\big(v' \ln u + \frac{vu'}{u}\big)$.

11. (a) See the answer in the text. (b) Let $f(x) = \ln(1 + x) - \frac{1}{2}x$. Then $f(0) = 0$ and moreover $f'(x) = 1/(x + 1) - \frac{1}{2} = (1 - x)/2(x + 1)$, which is positive in $(0, 1)$, so $f(x) > 0$ in $(0, 1)$, and the left-hand inequality is established. To prove the other inequality, put $g(x) = x - \ln(1 + x)$. Then $g(0) = 0$ and $g'(x) = 1 - 1/(x + 1) = x/(x + 1) > 0$ in $(0, 1)$, so the conclusion follows.
(c) Let $f(x) = 2(\sqrt{x} - 1) - \ln x$. Then $f(1) = 0$ and $f'(x) = (1/\sqrt{x}) - 1/x = (x - \sqrt{x})/x\sqrt{x} = (\sqrt{x} - 1)/x$, which is positive for $x > 1$. The conclusion follows.

Review Problems for Chapter 6

5. (a) $y = -3$ and $y' = -6x = -6$ at $x = 1$, so $y - (-3) = (-6)(x - 1)$, or $y = -6x + 3$.
(b) $y = -14$ and $y' = 1/2\sqrt{x} - 2x = -31/4$ at $x = 4$, so $y = -(31/4)x + 17$.
(c) $y = 0$ and $y' = (-2x^3 - 8x^2 + 6x)/(x + 3)^2 = -1/4$ at $x = 1$, so $y = (-1/4)(x - 1)$.

7. (a) $f(x) = x^3 + x$, etc. (b) Easy. (c) $h(y) = y(y^2 - 1) = y^3 - y$, etc. (d)–(f): use the quotient rule.

15. (a) $y' = \dfrac{2}{x}\ln x \geq 0$ if $x \geq 1$. (b) $y' = \dfrac{e^x - e^{-x}}{e^x + e^{-x}} \geq 0 \iff e^x \geq e^{-x} \iff e^{2x} \geq 0 \iff x \geq 0$
(c) $y' = 1 - \dfrac{3x}{x^2 + 2} = \dfrac{(x - 1)(x - 2)}{x^2 + 2} \geq 0 \iff x \leq 1$ or $x \geq 2$. (Use a sign diagram.)

Chapter 7 Derivatives in Use

7.1

2. Implicit differentiation yields $(*)$ $2xy + x^2(dy/dx) = 0$, and so $dy/dx = -2y/x$. Differentiating $(*)$ implicitly w.r.t. x gives $2y + 2x(dy/dx) + 2x(dy/dx) + x^2(d^2/dx^2) = 0$. Inserting the result for dy/dx, and simplifying yields $d^2/dx^2 = 6y/x^2$. (Alternatively, we can differentiate $-2y/x$ as a fraction.) These results follows more easily by differentiating $y = x^{-2}$ twice.

3. (a) Implicit differentiation w.r.t. x yields $(*)$ $1 - y' + 3y + 3xy' = 0$. Solving for y' yields $y' = (1 + 3y)/(1 - 3x) = -5/(1 - 3x)^2$. Differentiating $(*)$ w.r.t. x gives $-y'' + 3y' + 3y' + 3xy'' = 0$. Inserting $y' = (1 + 3y)/(1 - 3x)$ and solving for y'' gives $y'' = 6y'/(1 - 3x) = -30/(1 - 3x)^3$
(c) Implicit differentiation w.r.t. x yields $(*)$ $5y^4y' = 6x^5$, so $y' = 6x^5/5y^4 = (6/5)x^{1/5}$. Differentiating $(*)$ w.r.t. x gives $20y^3(y')^2 + 5y^4y'' = 30x^4$. Inserting $y' = 6x^5/5y^4$ and solving for y'' yields $y'' = 6x^4y^{-4} - (144/25)x^{10}y^{-9} = (6/25)x^{-4/5}$.

6. (a) $2x + 2yy' = 0$, and solve for y'. (b) $1/2\sqrt{x} + y'/2\sqrt{y} = 0$, and solve for y'.
(c) $4x^3 - 4y^3y' = 2xy^3 + x^2 3y^2 y'$, and solve for y'.

8. (a) $y + xy' = g'(x) + 3y^2y'$, and solve for y'. (b) $g'(x + y)(1 + y') = 2x + 2yy'$, and solve for y'.
(c) $2(xy + 1)(y + xy') = f'(x^2y)(2xy + x^2y')$, and solve for y'. (How did we differentiate $f(x^2y)$ w.r.t. x? Well, if $z = f(u)$ and $u = x^2y$, then $z' = f'(u)u'$ where u is a product of two functions that both depend on x. So $u' = 2xy + x^2y'$.)

10. (a) $2(x^2 + y^2)(2x + 2yy') = a^2(2x - 2yy')$, and solve for y'. (b) Note that $y' = 0$ when $x^2 + y^2 = a^2/2$, or $y^2 = \frac{1}{2}a^2 - x^2$. Inserting this into the given equation yields $x = \pm\frac{1}{4}a\sqrt{6}$. This yields the four points on the graph at which the tangent is horizontal.

7.2

1. Implicit differentiation w.r.t. P, with Q as a function of P, yields $\dfrac{dQ}{dP} \cdot P^{1/2} + Q\frac{1}{2}P^{-1/2} = 0$. Thus $\dfrac{dQ}{dP} = -\frac{1}{2}QP^{-1} = -\dfrac{19}{P^{3/2}}$.

5. Differentiating $(*)$ w.r.t. P yields $f''(P+t)\Big(\dfrac{dP}{dt} + 1\Big)^2 + f'(P+t)\dfrac{d^2P}{dt^2} = g''(P)\Big(\dfrac{dP}{dt}\Big)^2 + g'(P)\dfrac{d^2P}{dt^2}$. With simplified notation $f''(P' + 1)^2 + f'P'' = g''(P')^2 + g'P''$. Substituting $P' = f'/(g' - f')$ and solving for P'', we get $P'' = [f''(g')^2 - g''(f')^2]/(g' - f')^3$.

7.3

2. (a) $f'(x) = x^2\sqrt{4-x^2} + \dfrac{1}{3}x^3 \dfrac{-2x}{2\sqrt{4-x^2}} = \dfrac{4x^2(3-x^2)}{3\sqrt{4-x^2}}$. For the rest, see the answer in the text.

5. (a) See the text. (b) $dy/dx = -e^{-x}/(e^{-x}+3)$, so $dx/dy = -(e^{-x}+3)/e^{-x} = -1-3e^x$
(c) Implicit differentiation w.r.t. x yields $y^3 + x3y^2(dy/dx) - 3x^2y - x^3(dy/dx) = 2$. Solve for dy/dx, and then invert.

7.4

3. (a) $f(0) = 1$ and $f'(x) = -(1+x)^{-2}$, so $f'(0) = -1$. Then $f(x) \approx f(0) + f'(0)x = 1 - x$.
(b) $f(0) = 1$ and $f'(x) = 5(1+x)^4$, so $f'(0) = 5$. Then $f(x) \approx f(0) + f'(0)x = 1 + 5x$.
(c) $f(0) = 1$ and $f'(x) = -\frac{1}{4}(1-x)^{-3/4}$, so $f'(0) = -\frac{1}{4}$. Then $f(x) \approx f(0) + f'(0)x = 1 - \frac{1}{4}x$.

4. $F(1) = A$ and $F'(K) = \alpha AK^{\alpha-1}$, so $F'(1) = \alpha A$. Then $F(K) \approx F(1) + F'(1)(K-1) =$ $A + \alpha A(K-1) = A(1+\alpha A(K-1))$.

9. $3e^{xy^2} + 3xe^{xy^2}(y^2 + x2yy') - 2y' = 6x + 2yy'$. For $x = 1$, $y = 0$ this reduces to $3 - 2y' = 6$, so $y' = -3/2$. (b) $y(x) \approx y(1) + y'(1)(x-1) = -\frac{3}{2}(x-1)$

7.5

2. $f'(x) = (1+x)^{-1}$, $f''(x) = -(1+x)^{-2}$, $f'''(x) = 2(1+x)^{-3}$, $f^{(iv)}(x) = -6(1+x)^{-4}$, $f^{(v)}(x) = 24(1+x)^{-5}$. Then $f(0) = 0$, $f'(0) = 1$, $f''(0) = -1$, $f'''(0) = 2$, $f^{(iv)}(0) = -6$, $f^{(v)}(0) = 24$, and so $f(x) \approx f(0) + \frac{1}{1!}f'(0)x + \frac{1}{2!}f''(0)x + \frac{1}{3!}f'''(0)x^3 + \frac{1}{4!}f^{(iv)}(0)x^4 + \frac{1}{5!}f^{(v)}(0)x^5 = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5$

3. With $f(x) = 5(\ln(1+x) - \sqrt{1+x}\,) = 5\ln(1+x) - 5(1+x)^{1/2}$ we get $f'(x) = 5(1+x)^{-1} - \frac{5}{2}(1+x)^{-1/2}$, $f''(x) = -5(1+x)^{-2} + \frac{5}{4}(1+x)^{-3/2}$, and so $f(0) = -5$, $f'(0) = \frac{5}{2}$, $f''(0) = -\frac{15}{4}$. and the Taylor polynomial of degree 2 about $x = 0$ is $f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 = -5 + \frac{5}{2}x - \frac{15}{8}x^2$.

9. $h'(x) = \dfrac{(px^{p-1} - qx^{q-1})(x^p + x^q) - (x^p - x^q)(px^{p-1} + qx^{q-1})}{(x^p + x^q)^2} = \dfrac{2(p-q)x^{p+q-1}}{(x^p + x^q)^2}$, so $h'(1) = \frac{1}{2}(p-q)$. Since $h(1) = 0$, we get $h(x) \approx h(1) + h'(1)(x-1) = \frac{1}{2}(p-q)(x-1)$.

7.6

1. From Problem 7.5.2, $f(0) = 0$, $f'(0) = 1$, $f''(0) = -1$, and $f'''(c) = 2(1+c)^{-3}$. Then (3) gives $f(x) = f(0) + \frac{1}{1!}f'(0)x + \frac{1}{2!}f''(0)x + \frac{1}{3!}f'''(c)x^3 = x - \frac{1}{2}x^2 + \frac{1}{3}(1+c)^{-3}x^3$.

4. (a) With $g(x) = (1+x)^{1/3}$, $g'(x) = \frac{1}{3}(1+x)^{-2/3}$, $g''(x) = -\frac{2}{9}(1+x)^{-5/3}$, and $g'''(x) = \frac{10}{27}(1+x)^{-8/3}$, so $g(0) = 1$, $g'(0) = \frac{1}{3}$, $g''(0) = -\frac{2}{9}$, $g'''(c) = \frac{10}{27}(1+c)^{-8/3}$, so

$$g(x) = 1 + \tfrac{1}{3}x - \tfrac{1}{9}x^2 + R_3(x), \quad \text{where} \quad R_3(x) = \tfrac{1}{6!}\tfrac{10}{27}(1+c)^{-8/3}x^3 = \tfrac{5}{81}(1+c)^{-8/3}x^3$$

(b) $c \in (0, x)$ and $x \geq 0$, so $(1+c)^{-8/3} \leq 1$, and the inequality follows.
(c) $\sqrt[3]{1003} = 10(1+3\cdot 10^{-3})^{1/3} \approx 10.0099900$, using part (a) to approximate $(1+3\cdot 10^{-3})^{1/3}$. The error in (b) is $|R_3(x)| \leq \frac{5}{81}(3\cdot 10^{-3})^3 = \frac{5}{3}10^{-9}$. So the error in $\sqrt[3]{1003}$ is $\leq 10|R_3(x)| = \frac{50}{3}10^{-9} < 2\cdot 10^{-8}$, and the answer is correct to 7 decimal places.

7.7

4. (a) $\mathrm{El}_x e^{ax} = (x/e^{ax})ae^{ax} = ax$ (b) $\mathrm{El}_x \ln x = (x/\ln x)(1/x) = 1/\ln x$

(c) $\mathrm{El}_x(x^p e^{ax}) = \dfrac{x}{x^p e^{ax}}(px^{p-1}e^{ax} + x^p a e^{ax}) = p + ax$

(d) $\mathrm{El}_x(x^p \ln x) = \dfrac{x}{x^p \ln x}(px^{p-1}\ln x + x^p(1/x)) = p + 1/\ln x$

9. (a) $\mathrm{El}_x A = \dfrac{x}{A}\dfrac{dA}{dx} = 0$ (b) $\mathrm{El}_x(fg) = \dfrac{x}{fg}(fg)' = \dfrac{x}{fg}(f'g + fg') = \dfrac{xf'}{f} + \dfrac{xg'}{g} = \mathrm{El}_x f + \mathrm{El}_x g$

(c) $\mathrm{El}_x \dfrac{f}{g} = \dfrac{x}{(f/g)}\left(\dfrac{f}{g}\right)' = \dfrac{xg}{f}\left(\dfrac{gf' - fg'}{g^2}\right) = \dfrac{xf'}{f} - \dfrac{xg'}{g} = \mathrm{El}_x f - \mathrm{El}_x g$

(d) See the answer in the text. (e) Is like (d), but with $+g$ replaced by $-g$, and $+g'$ by $-g'$.

(f) $z = f(g(u))$, $u = g(x) \Rightarrow \mathrm{El}_x z = \dfrac{x}{z}\dfrac{dz}{dx} = \dfrac{x}{u}\dfrac{u}{z}\dfrac{dz}{du}\dfrac{du}{dx} = \mathrm{El}_u f(u)\,\mathrm{El}_x u$

7.8

3. Using (7.8.4), the functions are continuous wherever they are defined. So (a) and (d) are defined everywhere. In (b) we must exclude $x = 1$, in (c) the function is defined for $x < 2$, in (e) we must exclude $x = \pm\sqrt{3} - 1$, because the denominator is 0 for these values of x. Finally, in (f), the first fraction requires $x > 0$, and then the other fraction is also defined.

7.9

1. (b) $|x| = -x$ for $x < 0$. Hence, $\displaystyle\lim_{x\to 0^-} \frac{x + |x|}{x} = \lim_{x\to 0^-} \frac{x - x}{x} = \lim_{x\to 0^-} 0 = 0$.

(c) $|x| = x$ for $x > 0$. Hence, $\displaystyle\lim_{x\to 0^+} \frac{x + |x|}{x} = \lim_{x\to 0^+} \frac{x + x}{x} = \lim_{x\to 0^+} 2 = 2$.

(d) When $x \to 0^+$, $\sqrt{x} \to 0$, so $-1/\sqrt{x} \to -\infty$. (e) When $x \to 3^+$, $x - 3 \to 0^+$, and so $x/(x-3) \to \infty$. (f) When $x \to 3^-$, $x - 3 \to 0^-$, and so $x/(x-3) \to -\infty$.

4. (a) Vertical asymptote, $x = -1$. Moreover, $x^2 \div (x+1) = x - 1 + 1/(x+1)$, so $y = x - 1$ is an asymptote as $x \to \pm\infty$. (b) No vertical asymptote. Moreover. $(2x^3 - 3x^2 + 3x - 6) \div (x^2 + 1) = 2x - 3 + (x-3)/(x^2+1)$, so $y = 2x - 3$ is an asymptote as $x \to \pm\infty$. (c) Vertical asymptote, $x = 1$. Moreover, $(3x^2 + 2x) \div (x - 1) = 3x + 5 + 5/(x-1)$, so $y = 3x + 5$ is an asymptote as $x \to \pm\infty$. (d) Vertical asymptote, $x = 1$. Moreover, $(5x^4 - 3x^2 + 1) \div (x^3 - 1) = 5x + (-3x^2 + 5x + 1)/(x^3 - 1)$, so $y = 5x$ is an asymptote as $x \to \pm\infty$.

7.10

4. Recall from Note 4.7.2 that any integer root of the equation $f(x) = x^4 + 3x^3 - 3x^2 - 8x + 3 = 0$ must be a factor of the constant term 3. The way to see this directly is to notice that we must have

$$3 = -x^4 - 3x^3 + 3x^2 + 8x = x(-x^3 - 3x^2 + 3x + 8)$$

and if x is an integer then the bracketed expression is also an integer. Thus, the only possible integer solutions are ± 1 and ± 3. Trying each of these possibilities, we find that only -3 is an integer solution.

We are told in the problem that there are three other real roots, with approximate values $x_0 = -1.9$, $y_0 = 0.4$, and $z_0 = 1.5$. If we use Newton's method once for each of these roots we get new approximations

$$x_1 = -1.9 - \frac{f(-1.9)}{f'(-1.9)} = -1.9 - \frac{-0.1749}{8.454} \approx -1.9 + 0.021 = -1.879$$
$$y_1 = 0.4 - \frac{f(0.4)}{f'(0.4)} = 0.4 - \frac{-0.4624}{-8.704} \approx 0.4 - 0.053 = 0.347$$
$$z_1 = 1.5 - \frac{f(1.5)}{f'(1.5)} = 1.5 - \frac{-0.5625}{16.75} \approx 1.5 + 0.034 = 1.534$$

It can be shown by more precise calculations that the actual roots, rounded to six decimals, are -1.879385, 0.347296, and 1.532089.

7.11

2. (a) When $n \to \infty$, $2/n \to 0$ and so $5 - 2/n \to 5$. (b) When $n \to \infty$, $\dfrac{n^2-1}{n} = n - 1/n \to \infty$.
(c) When $n \to \infty$, $\dfrac{3n}{\sqrt{2n^2-1}} = \dfrac{3n}{n\sqrt{2-1/n^2}} = \dfrac{3}{\sqrt{2-1/n^2}} \to \dfrac{3}{\sqrt{2}} = \dfrac{3\sqrt{2}}{3}$.

7.12

2. L'Hôpital's rule yields $\lim\limits_{x\to a} \dfrac{x^2-a^2}{x-a} = \text{“}\dfrac{0}{0}\text{”} = \lim\limits_{x\to a} \dfrac{2x}{1} = 2a$. But note that we don't really need l'Hôpital's rule here, because $x^2 - a^2 = (x+a)(x-a)$, and therefore $\lim\limits_{x\to a} \dfrac{x^2-a^2}{x-a} = \lim\limits_{x\to a}(x+a) = 2a$.
(b) $\lim\limits_{x\to 0} \dfrac{2(1+x)^{1/2}-2-x}{2(1+x+x^2)^{1/2}-2-x} = \text{“}\dfrac{0}{0}\text{”} = \lim\limits_{x\to 0} \dfrac{(1+x)^{-1/2}-1}{(1+2x)(1+x+x^2)^{-1/2}-1} = \text{“}\dfrac{0}{0}\text{”} =$
$\lim\limits_{x\to 0} \dfrac{-\frac{1}{2}(1+x)^{-3/2}}{2(1+x+x^2)^{-1/2} + (1+2x)^2(-\frac{1}{2})(1+x+x^2)^{-3/2}} = -\dfrac{1}{3}$

7. $L = \lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to a} \dfrac{1/g(x)}{1/f(x)} = \text{“}\dfrac{0}{0}\text{”} = \lim\limits_{x\to a} \dfrac{-1/(g(x))^2}{-1/(f(x))^2} \cdot \dfrac{g'(x)}{f'(x)} = \lim\limits_{x\to a} \dfrac{(f(x))^2}{(g(x))^2} \cdot \dfrac{g'(x)}{f'(x)} =$
$L^2 \lim\limits_{x\to a} \dfrac{g'(x)}{f'(x)} = L^2 \lim\limits_{x\to a} \dfrac{1}{f'(x)/g'(x)}$. The conclusion follows. (Here, we have ignored problems with “division by 0”, when either $f'(x)$ or $g'(x)$ tends to 0 as x tends to a.)

Review Problems for Chapter 7

2. $5y^4y' - y^2 - 2xyy' = 0$, so $y' = \dfrac{y^2}{5y^4 - 2xy} = \dfrac{y}{5y^3 - 2x}$. Because $y = 0$ makes the given equation meaningless, y' is never 0.

6. $y' = 0$ when $1 + \frac{1}{5}\ln x = 0$, i.e. $\ln x = -5$, and then $x = e^{-5}$.

7. (a) We must have $\dfrac{1+x}{1-x} > 0$, i.e $-1 < x < 1$. When $x \to 1^-$, $f(x) \to \infty$. When $x \to -1^-$, $f(x) \to -\infty$. Since $f'(x) = 1/(1-x^2) > 0$ when $-1 < x < 1$, f is strictly increasing and the range of f is $\mathbb{R}$. (b) From $y = \dfrac{1}{2}\ln\dfrac{1+x}{1-x}$, $\ln\dfrac{1+x}{1-x} = 2y$, so $\dfrac{1+x}{1-x} = e^{2y}$. Then solve for x.

9. (a) $f(0) = \ln 4$ and $f'(x) = 2/(2x+4)$, so $f'(0) = 1/2$. Then $f(x) \approx f(0) + f'(0)x = \ln 4 + x/2$.
(b) $g(0) = 1$ and $g'(x) = -(1/2)(1+x)^{-3/2}$, so $g'(0) = -1/2$. Then $g(x) \approx g(0) + g'(0)x = 1 - x/2$.
(c) $h(0) = 0$ and $h'(x) = e^{2x} + 2xe^{2x}$, so $h'(0) = 1$. Then $h(x) \approx h(0) + h'(0)x = x$.

12. With $x = \frac{1}{2}$ and $n = 5$, formula (7.6.6) yields, $e^{\frac{1}{2}} = 1 + \frac{\frac{1}{2}}{1!} + \frac{(\frac{1}{2})^2}{2!} + \frac{(\frac{1}{2})^3}{3!} + \frac{(\frac{1}{2})^4}{4!} + \frac{(\frac{1}{2})^5}{5!} + \frac{(\frac{1}{2})^6}{6!}e^c$, where c is some number between 0 and $\frac{1}{2}$. Now, $R_6(\frac{1}{2}) = \frac{(\frac{1}{2})^6}{6!}e^c < \frac{(\frac{1}{2})^6}{6!}2 = \frac{1}{23040} \approx 0.0004340$, where we used the fact that since $c < \frac{1}{2}$, $e^c < e^{\frac{1}{2}} < 2$. Thus it follows that
$e^{\frac{1}{2}} \approx 1 + \frac{\frac{1}{2}}{1!} + \frac{(\frac{1}{2})^2}{2!} + \frac{(\frac{1}{2})^3}{3!} + \frac{(\frac{1}{2})^4}{4!} + \frac{(\frac{1}{2})^5}{5!} = 1 + \frac{1}{2} + \frac{1}{8} + \frac{1}{48} + \frac{1}{384} + \frac{1}{3840} \approx 1.6486979$. The error is less than 0.000043, and $e^{\frac{1}{2}} \approx 1.649$ correct to 3 decimals.

14. $y' + (1/y)y' = 1$, or $(*)$ $yy' + y' = y$. When $y = 1$, $y' = 1/2$. Differentiating $(*)$ w.r.t, x yields $(y')^2 + yy'' + y'' = y'$. With $y = 1$ and $y' = 1/2$, we find $y'' = 1/8$, so $y(x) \approx 1 + \frac{1}{2}x + \frac{1}{16}x^2$.

21. (a) $\lim\limits_{x\to 0} \dfrac{(2-x)e^x - x - 2}{x^3} = \text{“}\dfrac{0}{0}\text{”} = \lim\limits_{x\to 0} \dfrac{-e^x + (2-x)e^x - 1}{3x^2} = \text{“}\dfrac{0}{0}\text{”} = \lim\limits_{x\to 0} \dfrac{-e^x - e^x + (2-x)e^x}{6x} = \lim\limits_{x\to 0} \dfrac{-xe^x}{6x} = \lim\limits_{x\to 0} \dfrac{-e^x}{6} = -\dfrac{1}{6}$. (By canceling x, we needed to use l'Hôpital's rule only twice.)
(b) $\lim\limits_{x\to 3} \left(\dfrac{1}{x-3} - \dfrac{5}{x^2 - x - 6}\right) = \lim\limits_{x\to 3} \dfrac{x^2 - 6x + 9}{x^3 - 4x^2 - 3x + 18} = \text{“}\dfrac{0}{0}\text{”} = \lim\limits_{x\to 3} \dfrac{2x - 6}{3x^2 - 8x - 3} = \text{“}\dfrac{0}{0}\text{”} = \lim\limits_{x\to 3} \dfrac{2}{6x - 8} = \dfrac{1}{5}$ (c) $\lim\limits_{x\to 4} \dfrac{x-4}{2x^2 - 32} = \text{“}\dfrac{0}{0}\text{”} = \lim\limits_{x\to 4} \dfrac{1}{4x} = \dfrac{1}{16}$. (Can you find another way?)

23. (a) $\lim\limits_{x\to 1} \dfrac{\ln x - x + 1}{(x-1)^2} = \text{“}\dfrac{0}{0}\text{”} = \lim\limits_{x\to 1} \dfrac{(1/x) - 1}{2(x-1)} = \text{“}\dfrac{0}{0}\text{”} = \lim\limits_{x\to 1} \dfrac{(-1/x^2)}{2} = -\dfrac{1}{2}$
(b) $\lim\limits_{x\to 1} \dfrac{1}{x-1} \ln\left(\dfrac{7x+1}{4x+4}\right) = \lim\limits_{x\to 1} \dfrac{\ln(7x+1) - \ln(4x+4)}{x-1} = \text{“}\dfrac{0}{0}\text{”} = \lim\limits_{x\to 1} \dfrac{\dfrac{7}{7x+1} - \dfrac{4}{4x+4}}{1} = \dfrac{3}{8}$
(c) $\lim\limits_{x\to 1} \dfrac{x^x - x}{1 - x + \ln x} = \text{“}\dfrac{0}{0}\text{”} = \lim\limits_{x\to 1} \dfrac{x^x(\ln x + 1) - 1}{-1 + 1/x} = \text{“}\dfrac{0}{0}\text{”} = \lim\limits_{x\to 1} \dfrac{x^x(\ln x + 1)^2 + x^x(1/x)}{-1/x^2} = -2$
(using Example 6.11.4 to differentiate x^x).

24. When $x \to 0$, the denominator tends to $\sqrt{b} - \sqrt{d}$ and the numerator to 0, so the limit does not exist when $b \neq d$. If $b = d$, see the text.

Chapter 8 Single-Variable Optimization

8.1

1. (a) $f(0) = 2$ and $f(x) \le 2$ for all x (we divide 8 by a number larger than 2), so $x = 0$ maximizes $f(x)$. (b) $g(-2) = -3$ and $g(x) \ge -3$ for all x, so $x = -2$ minimizes $g(x)$. $g(x) \to \infty$ as $x \to \infty$, so there is no maximum. (c) $h(x)$ has its largest value 1 when $1 + x^2$ is the smallest, namely for $x = 0$, and $h(x)$ has its smallest value 1/2 when $1 + x^2$ is the largest, namely for $x = \pm 1$.

8.2

2. See the text, but note that the sign variation alone is not sufficient to conclude that the two stationary points are extreme points. It is important to point out that $h(x) \to 0$ as $x \to \pm\infty$. Sketch the graph. (For example, $f(x) = 3x - x^3$ is decreasing in $(-\infty, -1]$, increasing in $[-1, 1]$, and decreasing in $[1, \infty)$ but has no maximum or minimum, since $f(x) \to \infty$ as $x \to -\infty$, and $f(x) \to -\infty$ as $x \to \infty$. In fact, $x = -1$ is a local minimum point and $x = 1$ is a local maximum point.)

3. $h'(t) = 1/2\sqrt{t} - \frac{1}{2} = (1 - \sqrt{t})/2\sqrt{t}$. We see that $h'(t) \ge 0$ in $[0, 1]$ and $h'(t) \le 0$ in $[1, \infty)$. According to Theorem 8.2.1(a), $t = 1$ maximizes $h(t)$.

5. $f'(x) = 3x^2 \ln x + x^3/x = 3x^2(\ln x + \frac{1}{3})$. $f'(x) = 0$ when $\ln x = -\frac{1}{3}$, i.e. $x = e^{-1/3}$. We see that $f'(x) \le 0$ in $(0, e^{-1/3}]$ and $f'(x) \ge 0$ in $[e^{-1/3}, \infty)$, so $x = e^{-1/3}$ minimizes $f(x)$. Since $f(x) \to \infty$ as $x \to \infty$, there is no maximum.

8. (a) $y' = e^x - 2e^{-2x}$, $y'' = e^x + 4e^{-2x}$. We see that $y' = 0$ when $e^x = 2e^{-2x}$, or $e^{3x} = 2$, i.e. $x = \frac{1}{3}\ln 2$. Since $y'' > 0$ everywhere, this is minimum point. (b) $y' = -2(x-a) - 4(x-b) = 0$ when $x = \frac{1}{3}(a+2b)$. This is a maximum point since $y'' = -6$ for all x.
(c) $y' = 1/x - 5 = 0$ when $x = \frac{1}{5}$. This is a maximum point since $y'' = -1/x^2 < 0$ for all $x > 0$.

10. (a) $f'(x) = k - A\alpha e^{-\alpha x} = 0$ when $x_0 = (1/\alpha)\ln(A\alpha/k)$. Note that $x_0 > 0$ iff $A\alpha > k$. Moreover, $f'(x) < 0$ if $x < x_0$ and $f'(x) > 0$ if $x > x_0$, so x_0 solves the minimization problem.
(b) Substituting for A in the answer to (a) gives the expression for the optimal height x_0. Its value increases as p_0 (probability of flooding) or V (cost of flooding) increases, but decreases as δ (interest rate) or k (marginal construction costs) increases. The signs of these responses are obviously what an economist would expect. (Not only an economist, actually.)

8.3

2. (a) $\pi(Q) = Q(a-Q) - kQ = -Q^2 + (a-k)Q$, so $\pi'(Q) = -2Q + (a-k) = 0$ for $Q^* = \frac{1}{2}(a-k)$. This maximizes π because $\pi''(Q) < 0$. The monopoly profit is $\pi(Q^*) = -(\frac{1}{2}(a-k))^2 + (a-k)\frac{1}{2}(a-k) = \frac{1}{4}(a-k)^2$. (b) $d\pi(Q^*)/dk = -\frac{1}{2}(a-k) = -Q^*$, as in Example 3. (c) The new profit function is $\hat{\pi}(Q) = \pi(Q) + sQ = -Q^2 + (a-k)Q + sQ$. $\hat{\pi}'(Q) = -2Q + a - k + s = 0$ when $\hat{Q} = \frac{1}{2}(a-k+s)$. Now $\hat{Q} = \frac{1}{2}(a-k+s) = a-k$ provided $s = a-k$, which is the subsidy required to induce the monopolist to produce $a-k$ units. (The answer in the text has the wrong sign.)

5. $\overline{T}'(W) = a\dfrac{pb(bW+c)^{p-1}W - (bW+c)^p}{W^2} = a(bW+c)^{p-1}\dfrac{pbW - bW - c}{W^2}$, e.t.c.

8.4

2. In all cases the maximum and minimum exist by the extreme value theorem. Follow the recipe in (8.4.1).
(a) $f(x)$ is strictly decreasing so maximum is at $x = 0$, minimum at $x = 3$.
(b) $f(-1) = f(2) = 10$ and $f'(x) = 3x^2 - 3 = 0$ at $x = \pm 1$. $f(1) = 6$.
(c) $f(x) = x + 1/x$. $f(1/2) = f(2) = 5/2$ and $f'(x) = 1 - 1/x^2 = 0$ at $x = \pm 1$. $f(1) = 2$.
(d) $f(-1) = 4$, $f(\sqrt{5}) = 0$, and $f'(x) = 5x^2(x^2 - 3) = 0$ at $x = 0$ and $x = \sqrt{3}$. $f(0) = 0$, $f(\sqrt{3}) = -6\sqrt{3}$. (e) $f(0) = 0$, $f(3000) = 4.5 \cdot 10^9$, $f'(x) = 3(x^2 - 3000x + 2 \cdot 10^6 = 3(x-500)(x-2000)$. $f(1000) = 2.5 \cdot 10^9$, $f(2000) = 2 \cdot 10^9$.

4. (a) When there are $60 + x$ passengers, the charter company earns $800 - 10x$ from each, so they earn $\$(60+x)(800-10x)$. The sports club earns 1/10 of that amount. (b) See the text.

6. (a) $(f(2) - f(1))/(2-1) = (4-1)/1 = 3$ and $f'(x) = 2x$, so $2x^* = 3$, and thus $x^* = 3/2$.
(b) $(f(1) - f(0))/1 = -1$ and $f'(x) = -2x/\sqrt{1-x^2}$, so $2x^*/\sqrt{1-(x*)^2} = 1$, so $x^* = \sqrt{5}/5$. (From $2x^*/\sqrt{1-(x*)^2} = 1$, $\sqrt{1-(x*)^2} = 2x^*$, and then $1 - (x^*)^2 = 4(x^*)^2$. The positive solution is $x^* = \sqrt{5}/5$. (c) $(f(6) - f(2))/4 = -1/6$ and $f'(x) = -2/x^2$, so $-2/(x^*)^2 = -1/6$, so $x^* = \sqrt{12}$.
(d) $(f(4) - f(0))/4 = 1/4$ and $f'(x) = x/\sqrt{9+x^2}$, so $x/\sqrt{9+(x^*)^2} = 1/4$, so $x^* = \sqrt{3}$.

8.5

1. $\pi(Q) = 10Q - \frac{1}{1000}Q^2 - (5000 + 2Q) = 8Q - \frac{1}{1000}Q^2 - 5000$. Since $\pi'(Q) = 8 - \frac{1}{500}Q = 0$ for $Q = 4000$, and $\pi''(Q) = -\frac{1}{500} < 0$, $Q = 4000$ maximizes profits.

4. (i) $\pi(Q) = 1840Q - (2Q^2 + 40Q + 5000) = 1800Q - 2Q^2 - 5000$. Since $\pi'(Q) = 1800 - 4Q = 0$ for $Q = 450$, and $\pi''(Q) = -4 < 0$, $Q = 450$ maximizes profits.
(ii) $\pi(Q) = 2200Q - 2Q^2 - 5000$. Since $\pi'(Q) = 2200 - 4Q = 0$ for $Q = 550$, and $\pi''(Q) = -4 < 0$, $Q = 550$ maximizes profits..
(iii) $\pi(Q) = -2Q^2 - 100Q - 5000$ is negative for all $Q \geq 0$, so $Q = 0$ obviously maximizes profits.

6. $\pi'(Q) = P - abQ^{b-1} = 0$ when $Q^{b-1} = P/ab$, i.e. $Q = (P/ab)^{1/(b-1)}$. Moreover, $\pi''(Q) = -ab(b-1)Q^{b-2} < 0$ for all $Q > 0$, so this is a maximum point.

8.6

2. (a) Strictly decreasing, so no extreme points. (b) $f'(x) = 3x^2 - 3 = 0$ for $x = \pm 1$. With $f''(x) = 6x$, $f''(-1) = -6$ and $f''(1) = 6$, so $x = -1$ is a loc. maximum point, and $x = 1$ is a loc. minimum point. (c) $f'(x) = 1 - 1/x^2 = 0$ for $x = \pm 1$. With $f''(x) = 2/x^3$, $f''(-1) = -2$ and $f''(1) = 2$, so $x = -1$ is a local maximum point, and $x = 1$ is a local minimum point. (d)–(f): see the text.

3. (a) $f(x)$ is defined if and only if $x \neq 0$ and $x \geq -6$. $f(x) = 0$ at $x = -6$ and at $x = -2$. At any other point x in the domain, $f(x)$ has the same sign as $(x+2)/x$, so $f(x) > 0$ if $x \in (-6, -2)$ or $x \in (0, \infty)$.
(b) We first find the derivative of f:

$$f'(x) = -\frac{2}{x^2}\sqrt{x+6} + \frac{x+2}{x}\frac{1}{2\sqrt{x+6}} = \frac{-4x - 24 + x^2 + 2x}{2x^2\sqrt{x+6}} = \frac{x^2 - 2x - 24}{2x^2\sqrt{x+6}} = \frac{(x+4)(x-6)}{2x^2\sqrt{x+6}}$$

By means of a sign diagram we see that $f'(x) > 0$ if $-6 < x < -4$, $f'(x) < 0$ if $-4 < x < 0$, $f'(x) < 0$ if $0 < x < 6$, $f'(x) > 0$ if $6 < x$. It follows that f is strictly increasing in $[-6, -4]$, decreasing in $[-4, 0)$, decreasing in $(0, 6]$, and increasing $[6, \infty)$. It follows from the first-derivative test (Thm. 8.6.1) that f has two local minimum points, $x_1 = -6$ and $x_2 = 6$, and one local maximum point, $x_3 = -4$, with $f(-6) = 0$, $f(6) = \frac{4}{3}\sqrt{8} = 8\sqrt{2}/3$, and $f(-4) = \frac{1}{2}\sqrt{2}$.
(c) Since $\lim_{x\to 0}\sqrt{x+6} = 6 > 0$, while $\lim_{x\to 0^-}(1 + 2/x) = -\infty$ and $\lim_{x\to 0^+}(1 + 2/x) = \infty$, we see that $\lim_{x\to 0^-} f(x) = -\infty$ and $\lim_{x\to 0^+} f(x) = \infty$. Furthermore,

$$\lim_{x\to\infty} f'(x) = \lim_{x\to\infty}\left(\frac{x^2 - 2x - 24}{2x^2} \cdot \frac{1}{\sqrt{x+6}}\right) = \frac{1}{2} \cdot 0 = 0$$

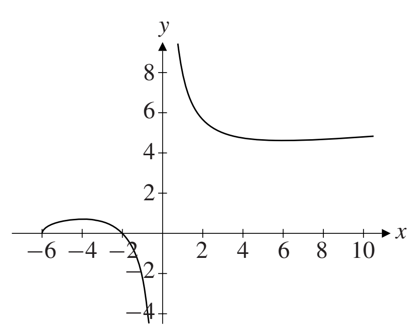


Figure SM8.6.3

4. Look at the point a. Since $f'(x)$ is graphed, $f'(x) < 0$ to the left of a, $f'(a) = 0$, and $f'(x) > 0$ to the right of a, so a is a local minimum point. At the points b and e, $f'(x) > 0$ on both sides of the points, so they cannot be extreme points.

6. (a) $f'(x) = x^2 e^x (3 + x)$. Use a sign diagram. ($x = 0$ is not an extreme point, but an inflection point.) (b) See the text and use a sign diagram for $g'(x)$, or check the sign of $g''(x) = 2^x(2+4x\ln x + x^2(\ln 2)^2)$ at the stationary points.

7. $f(x) = x^3 + ax + b \to \infty$ as $x \to \infty$, and $f(x) \to -\infty$ as $x \to -\infty$. Thus $f(x)$ has at least one real root. We have $f'(x) = 3x^2 + a$. Thus, for $a \geq 0$, $f'(x) > 0$ for all $x \neq 0$, so f is strictly increasing, and there is only one real root. Note that for $a \geq 0$, $4a^3 + 27b^2 \geq 0$. Assume next that $a < 0$. Then $f'(x) = 0$ for $x = \pm\sqrt{-a/3} = \pm\sqrt{p}$, where $p = -a/3 > 0$. Then f has a local maximum at $(-\sqrt{p}, b + 2p\sqrt{p})$ and a local minimum at $(\sqrt{p}, b - 2p\sqrt{p})$. If one of the local extreme values is 0, the equation has a double root, and this is the case iff $4p^3 = b^2$, that is, iff $4a^3 + 27b^2 = 0$. The equation has three real roots iff the local maximum value is positive and the local minimum value is negative. This occurs iff $|b| < 2p\sqrt{p}$ or iff $b^2 < 4p^3$ or iff $4a^3 + 27b^2 < 0$.

8.7

1. (a) $f'(x) = 3x^2 + 3x - 6 = 3(x-1)(x+2)$. Use a sign diagram and see the text. (b) $f''(x) = 6x + 3 = 0$ for $x = -1/2$ and $f''(x)$ changes sign around $x = -1/2$, so this is an inflection point.

3. Straightforward by using these derivatives: (a) $y' = -e^{-x}(1+x)$, $y'' = xe^{-x}$
(b) $y' = \dfrac{x-1}{x^2}$, $y'' = \dfrac{2-x}{x^3}$ (c) $y' = x^2 e^{-x}(3-x)$, $y'' = xe^{-x}(x^2 - 6x + 6)$
(d) $y' = \dfrac{1 - 2\ln x}{x^3}$, $y'' = \dfrac{6\ln x - 5}{x^4}$ (e) $y' = 2e^x(e^x - 1)$, $y'' = e^x(2e^x - 1)$
(f) $y' = 2e^{-x}(2 - x^2)$, $y'' = e^{-x}(x^2 - 2x - 2)$

Review Problems for Chapter 8

2. (a) $Q'(L) = 24L - \frac{3}{20}L^2 = 3L(8 - \frac{1}{20}L) = 0$ for $L^* = 160$, and $Q(L)$ is increasing in $[0, 160]$, decreasing in $[160, 200]$, so $Q^* = 160$ maximizes $Q(L)$. Output per worker is $Q(L)/L = 12L - \frac{1}{20}L^2$, and this quadratic function has maximum at $L^{**} = 120$. (b) See the text.

3. $\pi = -0.0016Q^2 + 44Q - 0.0004Q^2 - 8Q - 64\,000 = -0.002Q^2 + 36Q - 64\,000$, and $Q = 9000$ maximizes this quadratic function.
(b) $\text{El}_Q C(Q) = \dfrac{Q}{C(Q)} C'(Q) = \dfrac{0.0008Q^2 + 8Q}{0.0004Q^2 + 8Q + 64\,000} \approx 0.12$ when $Q = 1000$.

4. (a) See Problem 8.7.3(c). (b) $\lim_{x\to\infty} f(x) = 0$ according to (7.12.3), page 253 in the text. $\lim_{x\to-\infty} f(x) = -\infty$ because $x^3 \to -\infty$ and $e^{-x} \to \infty$. (See Fig. A8.R.4, page 671 in the text.)

5. (a) See the text. (b) A sign diagram shows that $f'(x) \geq 0$ in $(-1, 1]$ and $f'(x) \leq 0$ in $[1, \infty)$. Hence $x = 1$ is a maximum point. $f''(x) = \dfrac{-x(x^2 + x - 1)}{(x+1)^2} = 0$ for $x = 0$ and $x = \frac{1}{2}(\sqrt{5} - 1)$. ($x = \frac{1}{2}(-\sqrt{5} - 1)$ is outside the domain.) Since $f''(x)$ changes sign around these points, they are both inflection points.

6. (a) $h'(x) = \dfrac{e^x(2+e^{2x}) - e^x 2e^{2x}}{(2+e^{2x})^2} = \dfrac{e^x(2-e^{2x})}{(2+e^{2x})^2}$. See the text.

(b) h is strictly increasing in $(-\infty, 0]$, $\lim_{x\to-\infty} h(x) = 0$, and $h(0) = 1/3$. Thus, h defined on $(-\infty, 0]$ has an inverse defined on $(0, 1/3]$ with values in $(-\infty, 0]$. To find the inverse, note that $\dfrac{e^x}{2+e^{2x}} = y \iff y(e^x)^2 - e^x + 2y = 0$. This quadratic equation in e^x has the roots $e^x = [1 \pm \sqrt{1-8y^2}]/2y$. Since $y = 1/3$ for $x = 0$, see that we must have $e^x = [1-\sqrt{1-8y^2}]/2y$, and thus $x = \ln(1-\sqrt{1-8x^2}) - \ln(2x)$. Using x as the free variable, $h^{-1}(x) = \ln(1-\sqrt{1-8x^2}) - \ln(2x)$. The function and its inverse are graphed in Fig. SM8.R.6.

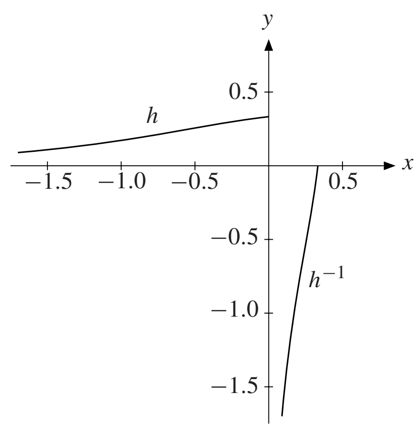


Figure SM8.R.6

8. $f'(x) = \dfrac{-6x^2(x^2-3)(x^2+2)}{(x^4+x^2+2)^2}$, so f is stationary when $x = 0$ and when $x = \pm\sqrt{3}$. $x = \sqrt{3}$ is a local (and global) maximum point, $x = -\sqrt{3}$ is a local (and global) minimum point, and $x = 0$ is neither. (It is an inflection point.) The graph of f is shown in Fig. A8.R.8 in the text, page 671.

9 Integration

9.1

1. This should be straightforward, since all the integrands are powers of x. Note that $x\sqrt{x} = x \cdot x^{1/2} = x^{3/2}$, $1/\sqrt{x} = x^{-1/2}$, and $\sqrt{x\sqrt{x\sqrt{x}}} = \sqrt{x\sqrt{x^{3/2}}} = \sqrt{x \cdot x^{3/4}} = \sqrt{x^{7/4}} = x^{7/8}$.

4. (a) $\int (t^3 + 2t - 3)\,dt = \int t^3\,dt + \int 2t\,dt - \int 3\,dt = \frac{1}{4}t^4 + t^2 - 3t + C$

(b) $\int (x-1)^2\,dx = \int (x^2-2x+1)\,dx = \frac{1}{3}x^3 - x^2 + x + C$. Alternative: Since $\dfrac{d}{dx}(x-1)^3 = 3(x-1)^2$, we have $\int (x-1)^2\,dx = \frac{1}{3}(x-1)^3 + C_1$. This agrees with the first answer, with $C_1 = C + 1/3$.

(c) $\int (x-1)(x+2)\,dx = \int (x^2+x-2)\,dx = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 2x + C$

(d) Either first evaluate $(x+2)^3 = x^3 + 6x^2 + 12x + 8$, to get $\int (x+2)^3\,dx = \frac{1}{4}x^4 + 2x^3 + 6x^2 + 8x + C$, or: $\int (x+2)^3 = \frac{1}{4}(x+2)^4 + C_1$. (e) $\int (e^{3x} - e^{2x} + e^x)\,dx = \frac{1}{3}e^{3x} - \frac{1}{2}e^{2x} + e^x + C$

(f) $\int \frac{x^3 - 3x + 4}{x}\,dx = \int \Big(x^2 - 3 + \frac{4}{x}\Big)dx = \frac{1}{3}x^3 - 3x + 4\ln|x| + C$

5. (a) First simplify the integrand: $\frac{(y-2)^2}{\sqrt{y}} = \frac{y^2 - 4y + 4}{\sqrt{y}} = y^{3/2} - 4y^{1/2} + 4y^{-1/2}$. From this we get $\int \frac{(y-2)^2}{\sqrt{y}} dy = \int (y^{3/2} - 4y^{1/2} + 4y^{-1/2})\,dy = \frac{2}{5}y^{5/2} - \frac{8}{3}y^{3/2} + 8y^{1/2} + C$.

(b) Polynomial division: $\frac{x^3}{x+1} = x^2 - x + 1 - \frac{1}{x+1}$, so $\int \frac{x^3}{x+1}\,dx = \frac{1}{3}x^3 - \frac{1}{2}x^2 + x - \ln|x+1| + C$.

(c) Since $\frac{d}{dx}(1+x^2)^{16} = 16(1+x^2)^{15} \cdot 2x = 32x(1+x^2)^{15}$, $\int x(1+x^2)^{15}\,dx = \frac{1}{32}(1+x^2)^{16} + C$.

10. (a) Easy. (b) (ii) $\sqrt{x+2} = (x+2)^{1/2}$, and use (a). (iii) $\frac{1}{\sqrt{4-x}} = (4-x)^{-1/2}$, and use (a).

11. $F(x) = \int(\frac{1}{2}e^x - 2x)\,dx = \frac{1}{2}e^x - x^2 + C$. $F(0) = \frac{1}{2}$ implies $C = 0$.
(b) $F(x) = \int(x - x^3)\,dx = \frac{1}{2}x^2 - \frac{1}{4}x^4 + C$. $F(1) = \frac{5}{12}$ implies $C = \frac{1}{6}$.

13. $f'(x) = \int(x^{-2} + x^3 + 2)\,dx = -x^{-1} + \frac{1}{4}x^4 + 2x + C$. With $f'(1) = 1/4$ we have $1/4 = -1 + \frac{1}{4} + 2 + C$, so $C = -1$. New integration yields $f(x) = \int(-x^{-1} + \frac{1}{4}x^4 + 2x - 1)\,dx = -\ln x + \frac{1}{20}x^5 + x^2 - x + D$. With $f(1) = 0$ we have $0 = -\ln 1 + \frac{1}{20} + 1 - 1 + D$, so $D = -1/20$.

9.2

5. We do only (c) and (f): $\int_{-2}^{3}\big(\frac{1}{2}x^2 - \frac{1}{3}x^3\big)\,dx = \Big|_{-2}^{3}(\frac{1}{6}x^3 - \frac{1}{12}x^4) = \Big|_{-2}^{3}(\frac{1}{12}x^3(2-x) = -\frac{27}{12} + \frac{32}{12} = \frac{5}{12}$.

(f) $\int_2^3 \Big(\frac{1}{t-1} + t\Big)\,dt = \Big|_2^3 \big[\ln(t-1) + \frac{1}{2}t^2\big] = \ln 2 + \frac{9}{2} - \frac{4}{2} = \ln 2 + \frac{5}{2}$

6. (a) A sign diagram shows that $f(x) > 0$ when $0 < x < 1$ and when $x > 2$. (b) $f(x) = x^3 - 3x^2 + 2x$, hence $f'(x) = 3x^2 - 6x + 2 = 0$ for $x_0 = 1 - \sqrt{3}/3$ and $x_1 = 1 + \sqrt{3}/3$. We see that $f'(x) > 0 \iff x < x_0$ or $x > x_1$. Also, $f'(x) < 0 \iff x_0 < x < x_1$. So f is (strictly) increasing in $(-\infty, x_0]$ and in $[x_1, \infty)$, and (strictly) decreasing in $[x_0, x_1]$. Hence x_0 is a local maximum point and x_1 is a local minimum point. (c) See the graph in the text. $\int_0^1 f(x)\,dx = \int_0^1(x^3 - 3x^2 + 2x)\,dx = \big|_0^1\big(\frac{x^4}{4} - x^3 + x^2\big) = \frac{1}{4} - 0 = \frac{1}{4}$.

7. (a) $f'(x) = -1 + \frac{3000\,000}{x^2} = 0$ for $x = \sqrt{3000\,000} = 1000\sqrt{3}$. (Recall $x > 0$.) For the rest, see the text.

9.3

2. (a) $\int_0^1 (x^{p+q} + x^{p+r})\,dx = \Big|_0^1 \frac{x^{p+q+1}}{p+q+1} + \frac{x^{p+r+1}}{p+r+1} = \frac{1}{p+q+1} + \frac{1}{p+r+1}$
(b) $f'(1) = 6$ implies $a + b = 6$. Since $f''(x) = 2ax + b$, $f''(1) = 18$ implies $2a + b = 18$. It follows that $a = 12$ and $b = 6$, so $f'(x) = 12x^2 - 6x$. But then $f(x) = \int(12x^2 - 6x)\,dx = 4x^3 - 3x^2 + C$, and since we want $\int_0^2(4x^3 - 3x^2 + C) = 18$, we must have $\big|_0^2(x^4 - x^3 + Cx) = 18$, i.e. $C = 5$.

3. (a) See text. (b) $\int_0^1 (x^2+2)^2\,dx = \int_0^1 (x^4 + 4x^2 + 4)\,dx = \Big|_0^1(\frac{1}{5}x^5 + \frac{4}{3}x^3 + 4x) = 83/15$

(c) $\int_0^1 \frac{x^2 + x + \sqrt{x+1}}{x+1}\,dx = \int_0^1 \frac{x(x+1) + (x+1)^{1/2}}{x+1}\,dx = \int_0^1 (x + (x+1)^{-1/2})\,dx =$

$\Big|_0^1 (\frac{1}{2}x^2 + 2(x+1)^{1/2})\,dx = 2\sqrt{2} - \frac{3}{2}$

(d) $A\dfrac{x+b}{x+c} + \dfrac{d}{x} = A\dfrac{x+c+b-c}{x+c} + \dfrac{d}{x} = A + \dfrac{A(b-c)}{x+c} + \dfrac{d}{x}$. Now integrate.

6. From $y^2 = 3x$ we get $x = \frac{1}{3}y^2$, which inserted into the other equation gives $y + 1 = (\frac{1}{3}y^2 - 1)^2$, or $y(y^3 - 6y - 9) = 0$. Here $y^3 - 6y - 9 = (y-3)(y^2 + 3y + 3)$, with $y^2 + 3y + 3$ never 0. So $(0, 0)$ and $(3, 3)$ are the only points of intersection. See the text.

7. $W(T) = K(1 - e^{-\varrho T})/\varrho T$. Here $W(T) \to 0$ as $T \to \infty$, and using l'Hôpital's rule, $W(T) \to K$ as $T \to 0^+$. For $T > 0$, we find $W'(T) = Ke^{-\varrho T}(1 + \varrho T - e^{\varrho T})/\varrho T^2 < 0$ because $e^{\varrho T} > 1 + \varrho T$ (see Problem 6.11.11). We conclude that $W(T)$ is strictly decreasing and that $W(T) \in (0, K)$.

8. (a) $f'(x) = \dfrac{2}{\sqrt{x+4}\,(\sqrt{x+4} - 2)} > 0$ for $x > 0$ and f has range $(-\infty, \infty)$, so f has an inverse defined on $(-\infty, \infty)$. We find that the inverse is $g(x) = e^{x/2} + 4e^{x/4}$. ($y = 4\ln(\sqrt{x+4} - 2) \iff \ln(\sqrt{x+4} - 2) = y/4 \iff \sqrt{x+4} = e^{y/4} + 2 \iff x + 4 = (e^{y/4} + 2)^2 \iff x = e^{y/2} + 4e^{y/4}$.) (b) See Fig. A9.3.8. (c) In Fig. A9.3.8 the graphs of f and g are symmetric about the line $y = x$, so area A = area B. But area B is the area of a rectangle with base a and height 10 minus the area below the graph of g over the interval $[0, a]$. Therefore, $B = 10a - \int_0^a (e^{x/2} + 4e^{x/4})\,dx = 10a + 18 - 2e^{a/2} - 16e^{a/4}$. Because $a = f(10) = 4\ln(\sqrt{14} - 2)$, this simplifies to $10a + 14 - 8\sqrt{14} \approx 6.26$.

9.4

2. (a) Let n be the total number of individuals. The number of individuals with income in the interval $[b, 2b]$ is then $N = n\int_b^{2b} Br^{-2}\,dr = n\Big|_b^{2b} -Br^{-1} = \dfrac{nB}{2b}$. Their total income is $M = n\int_b^{2b} Br^{-2}r\,dr = n\int_b^{2b} Br^{-1}\,dr = n\Big|_b^{2b} B\ln r = nB\ln 2$. Hence the mean income is $m = M/N = 2b\ln 2$.

(b) Total demand is $x(p) = \int_b^{2b} nD(p, r)f(r)\,dr = \int_b^{2b} nAp^\gamma r^\delta Br^{-2}\,dr = nABp^\gamma \int_b^{2b} r^{\delta-2}\,dr = nABp^\gamma \Big|_b^{2b} \dfrac{r^{\delta-1}}{\delta-1} = nABp^\gamma b^{\delta-1}\dfrac{2^{\delta-1} - 1}{\delta - 1}$.

5. (a) See Fig. A9.4.5. (b) $\int_0^t \big(g(\tau) - f(\tau)\big)\,d\tau = \int_0^t \big(2\tau^3 - 30\tau^2 + 100\tau\big)\,d\tau = \frac{1}{2}t^2(t - 10)^2 \geq 0$ for all t.

(c) $\int_0^{10} p(t)f(t) = \int_0^{10} \big(-t^3 + 9t^2 + 11t - 11 + 11/(t+1)\big)\,dt = 940 + 11\ln 11 \approx 966.38$,

$\int_0^{10} p(t)g(t)dt = \int_0^{10} \big(t^3 - 19t^2 + 79t + 121 - 121/(t+1)\big)\,dt = 3980/3 - 121\ln 11 \approx 1036.52$.

Profile g should be chosen.

9.5

1. (a) $\int \underset{\uparrow f}{x}\, \underset{\uparrow g'}{e^{-x}}\,dx = \underset{\uparrow f}{x}\,\underset{\uparrow g}{(-e^{-x})} - \int \underset{\uparrow f'}{1} \cdot \underset{\uparrow g}{(-e^{-x})}\,dx = -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x} + C$

(b) $\int 3xe^{4x}\,dx = 3x \cdot \frac{1}{4}e^{4x} - \int 3 \cdot \frac{1}{4}e^{4x}\,dx = \frac{3}{4}xe^{4x} - \frac{3}{16}e^{4x} + C$

(c) $\int (1+x^2)e^{-x}\,dx = (1+x^2)(-e^{-x}) - \int 2x(-e^{-x})\,dx = -(1+x^2)e^{-x} + 2\int xe^{-x}\,dx$
$= -(1+x^2)e^{-x} - 2xe^{-x} - 2e^{-x} + C = -(x^2+2x+3)e^{-x} + C$
(d) $\int x\ln x\,dx = \frac{1}{2}x^2\ln x - \int \frac{1}{2}x^2\frac{1}{x}\,dx = \frac{1}{2}x^2\ln x - \int \frac{1}{2}x\,dx = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C$

2. (a) See the text.
(b) Recall that $\frac{d}{dx}2^x = 2^x\ln 2$, and therefore $2^x/\ln 2$ is the indefinite integral of 2^x. If follows that
$\int_0^2 x2^x\,dx = \Big|_0^2 x\frac{2^x}{\ln 2} - \int_0^2 \frac{2^x}{\ln 2}\,dx = \frac{8}{\ln 2} - \Big|_0^2 \frac{2^x}{(\ln 2)^2} = \frac{8}{\ln 2} - \Big(\frac{4}{(\ln 2)^2} - \frac{1}{(\ln 2)^2}\Big) = \frac{8}{\ln 2} - \frac{3}{(\ln 2)^2}$
(c) First use integration by parts on the indefinite integral: with $f(x) = x^2$ and $g(x) = e^x$,
$(*)\ \int x^2e^x\,dx = x^2e^x - \int 2xe^x\,dx$. To evaluate the last integral we must use integration by parts once more: with $f(x) = 2x$ and $g(x) = e^x$, $\int 2xe^x\,dx = 2xe^x - \int 2e^x\,dx = 2xe^x - (2e^x + C)$. Inserted into $(*)$ this gives $\int x^2e^x\,dx = x^2e^x - 2xe^x + 2e^x + C$, and hence, $\int_0^1 x^2e^x\,dx = \Big|_0^1 (x^2e^x - 2xe^x + 2e^x) = (e - 2e + 2e) - (0 - 0 + 2) = e - 2$. Alternatively, more compactly using formula (9.5.2):
$\int_0^1 x^2e^x\,dx = \Big|_0^1 x^2e^x - 2\int_0^1 xe^x\,dx = e - 2\Big[\Big|_0^1 xe^x - \int_0^1 e^x\,dx\Big] = e - 2[e - \big|_0^1 e^x] = e - 2$

5. (a) By formula (9.5.2), $\int_0^T te^{-rt}\,dt = \Big|_0^T t\frac{-1}{r}e^{-rt} - \int_0^T \frac{-1}{r}e^{-rt}\,dt = \frac{-T}{r}e^{-rT} + \frac{1}{r}\int_0^T e^{-rt}\,dt$
$= \frac{-T}{r}e^{-rT} + \frac{1}{r}\Big|_0^T \frac{-1}{r}e^{-rt} = \frac{1}{r^2}(1 - (1+rT)e^{-rT})$. Multiply this expression by b.
(b) $\int_0^T (a+bt)e^{-rt}\,dt = a\int_0^T e^{-rt}\,dt + b\int_0^T te^{-rt}\,dt$, a.s.o. using (a).
(c) $\int_0^T (a - bt + ct^2)e^{-rt}\,dt = a\int_0^T e^{-rt}\,dt - b\int_0^T te^{-rt}\,dt + c\int_0^T t^2e^{-rt}\,dt$. Use the previous results and $\int_0^T t^2e^{-rt}\,dt = \Big|_0^T t^2(-1/r)e^{-rt} - \int_0^T 2t(-1/r)e^{-rt}\,dt = -(1/r)T^2e^{-rT} + (2/r)\int_0^T te^{-rt}\,dt$.

9.6

2. (a) See the text. (b) With $u = x^3 + 2$ we get $du = 3x^2\,dx$ and
$\int x^2e^{x^3+2}\,dx = \int \frac{1}{3}e^u\,du = \frac{1}{3}e^u + C = \frac{1}{3}e^{x^3+2} + C$.
(c) First attempt: $u = x + 2$, which gives $du = dx$ and $\int \frac{\ln(x+2)}{2x+4}\,dx = \int \frac{\ln u}{2u}\,du$.
This does not look very much simpler than the original integral. A better idea is to substitute $u = \ln(x+2)$. Then $du = \frac{dx}{x+2}$ and $\int \frac{\ln(x+2)}{2x+4}\,dx = \int \frac{1}{2}u\,du = \frac{1}{4}(u)^2 + C = \frac{1}{4}(\ln(x+2))^2 + C$.
(d) First attempt: $u = 1 + x$. Then, $du = dx$, and $\int x\sqrt{1+x}\,dx = \int (u-1)\sqrt{u}\,du = \int (u^{3/2} - u^{1/2})\,du$ $= \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C = \frac{2}{5}(1+x)^{5/2} - \frac{2}{3}(1+x)^{3/2} + C$. Second attempt: $u = \sqrt{1+x}$. Then $u^2 = 1 + x$ and $2u\,du = dx$. Then the integral is $\int x\sqrt{1+x}\,dx = \int (u^2-1)u2u\,du = \int (2u^4 - 2u^3)\,du$ e.t.c. Check that you get the same answer. Actually, even integration by parts works in this case. Put $f(x) = x$ and $g'(x) = \sqrt{1+x}$, and choose $g(x) = \frac{2}{3}(1+x)^{3/2}$. (The answer looks different, but is not.)

(e) With $u = 1 + x^2$, $x^2 = u - 1$, and $du = 2x dx$, so $\int \frac{x^3}{(1+x^2)^3}\,dx = \int \frac{x^2 \cdot x}{(1+x^2)^3}\,dx = \frac{1}{2}\int \frac{u-1}{u^3}\,du = \frac{1}{2}\int (u^{-2} - u^{-3})\,du = -\frac{1}{2}u^{-1} + \frac{1}{4}u^{-2} + C = \frac{-1}{2(1+x^2)} + \frac{1}{4(1+x^2)^2} + C.$

(f) With $u = \sqrt{4 - x^3}$, $u^2 = 4 - x^3$, and $2u du = -3x^2 dx$, so $\int x^5\sqrt{4-x^3}\,dx = \int x^3\sqrt{4-x^3}\,x^2\,dx = \int (4-u^2)\,u\,(-\frac{2}{3})u\,du = \int (-\frac{8}{3}u^2 + \frac{2}{3}u^4)\,du = -\frac{8}{9}u^3 + \frac{2}{15}u^5 + C = -\frac{8}{9}(4-x^3)^{3/2} + \frac{2}{15}(4-x^3)^{5/2} + C$

6. (a) $I = \int_0^1 (x^4 - x^9)(x^5 - 1)^{12}\,dx = \int_0^1 -x^4(x^5-1)^{13}\,dx$. Introduce $u = x^5 - 1$. Then $du = 5x^4 dx$, and when $x = 0$, $u = -1$, when $x = 1$, $u = 0$, and thus $I = -\int_{-1}^0 \frac{1}{5}u^{13}\,du = -\Big|_{-1}^0 \frac{1}{70}u^{14} = \frac{1}{70}$.

(b) With $u = \sqrt{x}$, $u^2 = x$, and $2u du = dx$, and so
$\int \frac{\ln x}{\sqrt{x}}\,dx = 2\int \ln u^2\,du = 4\int \ln u\,du = 4(u\ln u - u) + C = 4\sqrt{x}\ln\sqrt{x} - 4\sqrt{x} + C = 2\sqrt{x}\ln x - 4\sqrt{x} + C$. (Integration by parts also works in this case with $f(x) = \ln x$ and $g'(x) = 1/\sqrt{x}$.)

(c) With $u = 1 + \sqrt{x}$, $u - 1 = \sqrt{x}$, or $(u-1)^2 = x$, so $2(u-1)du = dx$. When $x = 0$, $u = 1$, when $x = 4$, $u = 3$. Hence,
$\int_0^4 \frac{dx}{\sqrt{1+\sqrt{x}}} = \int_1^3 \frac{2(u-1)}{\sqrt{u}}\,du = 2\int_1^3 (u^{1/2} - u^{-1/2})\,du = 2\Big|_1^3 (\frac{2}{3}(u^{3/2} - 2u^{1/2})\,du = \frac{8}{3}$
(The substitution $u = \sqrt{1+\sqrt{x}}$ also works.)

7. (a) With $u = 1 + e^{\sqrt{x}}$, $u > 0$, and $du = \frac{1}{2\sqrt{x}} \cdot e^{\sqrt{x}}\,dx$. When $x = 1$, $u = 1 + e$ and $x = 4$ gives $u = 1 + e^2$. Thus we get (note how we carry over the limits of integration):
$\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}\,(1+e^{\sqrt{x}})}\,dx = \int_{1+e}^{1+e^2} \frac{2\,du}{u} = 2\Big|_{1+e}^{1+e^2} \ln u = 2\ln(1+e^2) - 2\ln(1+e)$

(b) A natural substitution is $u = e^x + 1$, $du = e^x\,dx$, and $dx = du/e^x = du/(u-1)$. When $x = 0$, $u = 2$, when $x = 1/3$, $u = e^{1/3} + 1$. Thus, $\int_0^{1/3} \frac{dx}{e^x+1} = \int_2^{e^{1/3}+1} \frac{1}{u(u-1)}\,du = \int_2^{e^{1/3}+1} \Big(\frac{1}{u-1} - \frac{1}{u}\Big)\,du = \Big|_2^{e^{1/3}+1} \big(\ln|u-1| - \ln|u|\big) = \frac{1}{3} - \ln(e^{1/3}+1) + \ln 2 = \ln 2 - \ln(e^{-1/3}+1)$. (Verify the last equality.)
Rewriting the integrand as $\frac{e^{-x}}{1+e^{-x}}$, the suggested substitution $t = e^{-x}$ (or even better $u = 1 + e^{-x}$), $dt = -e^{-x}dx$ works well. Verify that you get the same answer.

9.7

3. (a) See answer in the text. Using a simplified notation and the result in Example 1(a), we have:
(b) $\int_0^\infty (x - 1/\lambda)^2\,\lambda e^{-\lambda x}\,dx = -\Big|_0^\infty (x-1/\lambda)^2\,e^{-\lambda x} + \int_0^\infty 2\,(x - 1/\lambda)\,e^{-\lambda x}\,dx$
$= 1/\lambda^2 + 2\int_0^\infty x e^{-\lambda x}\,dx - (2/\lambda)\int_0^\infty e^{-\lambda x}\,dx = 1/\lambda^2 + 2/\lambda^2 - 2/\lambda^2 = 1/\lambda^2$
(c) $\int_0^\infty (x - 1/\lambda)^3\,\lambda e^{-\lambda x}\,dx = -\Big|_0^\infty (x-1/\lambda)^3\,e^{-\lambda x} + \int_0^\infty 3\,(x-1/\lambda)^2\,e^{-\lambda x}\,dx$
$= -1/\lambda^3 + (3/\lambda)\int_0^\infty (x-1/\lambda)^2\,\lambda e^{-\lambda x}\,dx = -1/\lambda^3 + (3/\lambda)(1/\lambda^2) = 2/\lambda^3$

5. (a) $f'(x) = (1 - 3\ln x)/x^4 = 0$ at $x = e^{1/3}$, and $f'(x) > 0$ for $x < e^{1/3}$ and $f'(x) < 0$ for $x > e^{1/3}$. Hence f has a maximum at $(e^{1/3}, 1/3e)$. Since $f(x) \to -\infty$ as $x \to 0^+$, there is no minimum. Note that $f(x) \to 0$ as $x \to \infty$. (Use l'Hôpital's rule.)

(b) $\int_a^b x^{-3}\ln x\,dx = -\Big|_a^b \frac{1}{2}x^{-2}\ln x + \int_a^b \frac{1}{2}x^{-3}\,dx = \Big|_a^b(-\frac{1}{2}x^{-2}\ln x - \frac{1}{4}x^{-2})$. This diverges when $b = 1$ and $a \to 0$. But $\int_1^\infty x^{-3}\ln x\,dx = 1/4$.

7. If both limits exist, the integral is the sum of the following two limits: $I_1 = \lim_{\varepsilon\to 0^+}\int_{-2+\varepsilon}^{3}\big(1/\sqrt{x+2}\big)\,dx$ and $I_2 = \lim_{\varepsilon\to 0^+}\int_{-2}^{3-\varepsilon}\big(1/\sqrt{3-x}\big)\,dx$. Here $I_1 = \lim\limits_{\varepsilon\to 0^+}\Big|_{-2+\varepsilon}^{3}\big(2\sqrt{x+2}\big) = \lim\limits_{\varepsilon\to 0^+}\big(2\sqrt{5}-2\sqrt{\varepsilon}\big) = 2\sqrt{5}$, and $I_2 = \lim\limits_{\varepsilon\to 0^+}\Big|_{-2}^{3-\varepsilon}\big(-2\sqrt{3-x}\big) = \lim\limits_{\varepsilon\to 0^+}\big(-2\sqrt{\varepsilon}+2\sqrt{5}\big) = 2\sqrt{5}$.

12. (a) The suggested substitution gives $\displaystyle\int_{-\infty}^{+\infty} f(x)\,dx = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-u^2}\,du = 1$, by (9.7.8).

(b) Here $\displaystyle\int_{-\infty}^{+\infty} xf(x)\,dx = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}(\mu+\sqrt{2}\sigma u)e^{-u^2}\,du = \mu$, using part (a) and Example 3.

(c) Here $\displaystyle I = \int_{-\infty}^{+\infty} x^2 f(x)\,dx = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}(2\sigma^2u^2 + 2\sqrt{2}\sigma\mu u + \mu^2)e^{-u^2}\,du$

$\displaystyle = \frac{2\sigma^2}{\sqrt{\pi}}\int_{-\infty}^{+\infty} u^2e^{-u^2}\,du + \frac{2\sqrt{2}\sigma\mu}{\sqrt{\pi}}\int_{-\infty}^{+\infty} ue^{-u^2}\,du + \frac{\mu^2}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-u^2}\,du = \sigma^2 + 0 + \mu^2$. (Note how integration by parts gives $\displaystyle\int u^2e^{-u^2}\,du = -\tfrac{1}{2}ue^{-u^2} + \int \tfrac{1}{2}e^{-u^2}\,du$, so $\displaystyle\int_{-\infty}^{+\infty} u^2e^{-u^2}\,du = \tfrac{1}{2}\sqrt{\pi}$.)

9.8

5. $P(10) = 705$ gives $641e^{10k} = 705$, or $e^{10k} = 705/641$. Taking the natural logarithm of both sides yields $10k = \ln(705/641)$, so $k = 0.1\ln(705/641)$.

7. Straightforward. Note that in (9.8.10), if $b \neq 0$, there are always two constant solutions, $x \equiv 0$ and $x \equiv a/b$. The latter is obtained by letting $A = 0$ in (9.8.10). So in addition to the answer in the text, in (e) add $x \equiv 0$ as a solution, in (f) add $K \equiv 0$.

9. (a) Use (9.8.7). (Using (9.8.10), and then using $N(0) = 1$ to determine the constant, is less efficient.)
(b) $\displaystyle 800 = \frac{1000}{1+999e^{-0.39t^*}} \iff 999e^{-0.39t^*} = \frac{1}{4}$, so $e^{-0.39t^*} = 1/3996$, and so $0.39t^* = \ln 3996$, etc.

9.9

2. (a) $dx/dt = e^{2t}/x^2$. Separate: $\int x^2\,dx = \int e^{2t}\,dt$. Integrate: $\frac{1}{3}x^3 = \frac{1}{2}e^{2t} + C_1$. Solve for x: $x^3 = \frac{3}{2}e^{2t} + 3C_1 = \frac{3}{2}e^{2t} + C$, with $C = 3C_1$. Hence, $x = \sqrt[3]{\frac{3}{2}e^{2t} + C}$. (You cannot wait to put the constant "in the end". Wrong: $\frac{1}{3}x^3 = \frac{1}{2}e^{2t}$, $x^3 = \frac{3}{2}e^{2t}$, $x = \sqrt[3]{\frac{3}{2}e^{2t}} + C$. This is not a solution!)
(b) $\int e^{-x}\,dx = \int e^{-t}dt$. Integrate: $-e^{-x} = -e^{-t} + C_1$. Solve for x: $e^{-x} = e^{-t} + C$, with $C = -C_1$. Hence, $-x = \ln(e^{-t} + C)$, so $x = -\ln(e^{-t} + C)$. (c) Directly from (9.9.3). (d) Similar to (a).
(e) By (9.9.5), $x = Ce^{2t} + e^{2t}\int(-t)e^{-2t}\,dt = Ce^{2t} - e^{2t}\int te^{-2t}\,dt$. Here $\int te^{-2t}\,dt = t(-\frac{1}{2})e^{-2t} + \frac{1}{2}\int e^{-2t}\,dt = (-\frac{1}{2}t - \frac{1}{4})e^{-2t}$ and thus $x = Ce^{2t} - e^{2t}(-\frac{1}{2}t - \frac{1}{4})e^{-2t} = Ce^{2t} + \frac{1}{2}t + \frac{1}{4}$.
(f) Formula (9.9.5): $x = Ce^{-3t} + e^{-3t}\int e^{3t}te^{t^2-3t}\,dt = Ce^{-3t} + e^{-3t}\int te^{t^2}\,dt = Ce^{-3t} + \frac{1}{2}e^{t^2-3t}$

3. The equation is separable: $dk/k = s\alpha e^{\beta t}\,dt$, so $\ln k = \frac{s\alpha}{\beta}e^{\beta t} + C_1$, or $k = e^{\frac{s\alpha}{\beta}e^{\beta t}}e^{C_1} = Ce^{\frac{s\alpha}{\beta}e^{\beta t}}$. With $k(0) = k_0$, we have $k_0 = Ce^{\frac{s\alpha}{\beta}}$, and thus $k = k_0e^{\frac{s\alpha}{\beta}(e^{\beta t}-1)}$.

5. (a) See the text. (b) $\int K^{-\alpha}\,dK = \int \gamma L_0 e^{\beta t}\,dt$, so $\dfrac{1}{1-\alpha}K^{1-\alpha} = \dfrac{\gamma L_0}{\beta}e^{\beta t} + C_1$.
Hence, $K^{1-\alpha} = \dfrac{\gamma L_0(1-\alpha)}{\beta}e^{\beta t} + (1-\alpha)C_1$. For $t=0$, $K_0^{1-\alpha} = \dfrac{\gamma L_0(1-\alpha)}{\beta} + (1-\alpha)C_1$, so $K^{1-\alpha} = \dfrac{(1-\alpha)\gamma L_0}{\beta}(e^{\beta t}-1) + K_0^{1-\alpha}$, from which we find K.

6. $\dfrac{t}{x}\dfrac{dx}{dt} = a$ is separable: $\dfrac{dx}{x} = a\dfrac{dt}{t}$, so $\displaystyle\int \frac{dx}{x} = a\int \frac{dt}{t}$. Integrating yields, $\ln x = a\ln t + C_1$, so $x = e^{a\ln t + C_1} = (e^{\ln t})^a e^{C_1} = Ct^a$, with $C = e^{C_1}$.

Review Problems for Chapter 9

3. (a) $\displaystyle\int_0^{12} 50\,dx = \Big|_0^{12} 50x = 600$ (b) $\displaystyle\int_0^2 (x - \tfrac{1}{2}x^2)\,dx = \Big|_0^2 (\tfrac{1}{2}x^2 - \tfrac{1}{6}x^3) = \tfrac{2}{3}$
(c) $\displaystyle\int_{-3}^{3} (u+1)^2\,du = \Big|_{-3}^{3} \tfrac{1}{3}(u+1)^3\,du = 24$ (d) $\displaystyle\int_1^5 \frac{2}{z}\,dz = \Big|_1^5 2\ln z = 2\ln 5$
(e) $\displaystyle\int_2^{12} \frac{3\,dt}{t+4}\,dt = \Big|_2^{12} 3\ln(t+4) = 3\ln(8/3)$ (f) $\displaystyle\int_0^4 v\sqrt{v^2+9}\,dv = \Big|_0^4 \tfrac{1}{3}(v^2+9)^{3/2} = 98/3$

5. (a) With $u = 9 + \sqrt{x}$, $x = (u-9)^2$ and $dx = 2(u-9)\,du$. When $x=0$, $u=9$ and $x=25$ gives $u=14$. Thus, $\displaystyle\int_0^{25} \frac{1}{9+\sqrt{x}}\,dx = \int_9^{14} \frac{2(u-9)}{u}\,du = \int_9^{14}\Big(2 - \frac{18}{u}\Big)\,du = 10 - 18\ln\frac{14}{9}$.
(b) With $u = \sqrt{t+2}$, $t = u^2 - 2$, and $dt = 2u\,du$. When $t=2$, $u=2$, and $t=7$ gives $u=3$. Hence,
$\displaystyle\int_2^7 t\sqrt{t+2}\,dt = \int_2^3 (u^2-2)u\cdot 2u\,du = 2\int_2^3 (u^4 - 2u^2)\,du = 2\Big|_2^3\Big(\tfrac{1}{5}u^5 - \tfrac{2}{3}u^3\Big) = 886/15$
(c) With $u = \sqrt[3]{19x^3+8}$, $u^3 = 19x^3 + 8$, so $3u^2du = 57x^2dx$. When $x=0$, $u=2$ and $x=1$ gives $u=3$. Then $\displaystyle\int_0^1 57x^2\sqrt[3]{19x^3+8}\;dx = \int_2^3 3u^3\,du = \Big|_2^3 \tfrac{3}{4}u^4 = 195/4$.

10. Equilibrium when $50/(Q^*+5) = 10 + Q^*$, i.e. $(Q^*)^2 + 50Q^* - 275 = 0$. The only positive solution is $Q^* = 5$, and then $P^* = 5$. CS $= \displaystyle\int_0^5 \Big[\frac{50}{Q+5} - 5\Big]\,dQ = \Big|_0^5 [50\ln(Q+5) - 5Q] = 50\ln 2 - 25$,
PS $= \displaystyle\int_0^5 (5 - 4.5 - 0.1Q)\,dQ = 1.25$.

11. (a) $f'(t) = 4\dfrac{2\ln t\cdot(1/t)\cdot t - (\ln t)^2\cdot 1}{t^2} = 4\dfrac{(2-\ln t)\ln t}{t^2}$, and
$f''(t) = 4\dfrac{(2\cdot(1/t) - 2\ln t\cdot(1/t))\,t^2 - (2\ln t - (\ln t)^2)\,2t}{t^4} = 8\dfrac{(\ln t)^2 - 3\ln t + 1}{t^3}$
(b) Stationary points: $f'(t) = 0 \iff \ln t(2-\ln t) = 0 \iff \ln t = 2$ or $\ln t = 0 \iff t = e^2$ or $t=1$ Since $f''(1) = 8 > 0$ and $f''(e^2) = -8e^{-6}$, $t=1$ is a local minimum point and $t = e^2 \approx 7.4$ is a local maximum point. We find $f(1) = 0$ and $f(e^2) = 16e^{-2} \approx 2.2$.
(c) $\dfrac{d}{dt}\Big(\dfrac{4}{3}(\ln t)^3\Big) = \dfrac{4}{3}3(\ln t)^2\dfrac{1}{t} = f(t)$, so $\displaystyle\int f(t)\,dt = \frac{4}{3}(\ln t)^3 + C$. Since $f(t) \geq 0$ for all $t > 0$, the area is $\displaystyle\int_1^{e^2} f(t)\,dt = \Big|_1^{e^2}\frac{4}{3}(\ln t)^3 = \frac{4}{3}2^3 - 0 = \frac{32}{3}$.

12. Straightforward by using (9.8.8)–(9.8.10).

13. (a) Separable. $\int x^{-2}\,dx = \int t\,dt$, and so $-1/x = \frac{1}{2}t^2 + C_1$, or $x = 1/(C - \frac{1}{2}t^2)$, (with $C = -C_1$). (b) and (c): Direct use of (9.9.3). (d) Using (9.9.5), $x = Ce^{-5t} + 10e^{-5t}\int te^{5t}\,dt$. Here $\int te^{5t}\,dt = t\frac{1}{5}e^{5t} - \frac{1}{5}\int e^{5t}\,dt = \frac{1}{5}te^{5t} - \frac{1}{25}e^{5t}$. Thus $x = Ce^{-5t} + 10e^{-5t}(\frac{1}{5}te^{5t} - \frac{1}{25}e^{5t}) = Ce^{-5t} + 2t - \frac{2}{5}$.
(e) $x = Ce^{-t/2} + e^{-t/2}\int e^{t/2}e^{t}\,dt = Ce^{-t/2} + e^{-t/2}\int e^{3t/2}\,dt = Ce^{-t/2} + e^{-t/2}\frac{2}{3}e^{3t/2} = Ce^{-t/2} + \frac{2}{3}e^{t}$.
(e) $x = Ce^{-3t} + e^{-3t}\int t^2e^{3t}\,dt = Ce^{-3t} + e^{-3t}(\frac{1}{3}t^2e^{3t} - \frac{2}{3}\int te^{3t}\,dt)$
$= Ce^{-3t} + \frac{1}{3}t^2 - \frac{2}{3}e^{-3t}(\frac{1}{3}te^{3t} - \frac{1}{3}\int e^{3t}\,dt) = Ce^{-3t} + \frac{1}{3}t^2 - \frac{2}{9}t + \frac{2}{27}$.

16. (a) and (b), see the text. (c) $F''(x) = f'(x) = a\lambda^2e^{-\lambda x}(e^{-\lambda x} - a)(e^{-\lambda x} + a)^{-3}$. Note that $F''(x) = 0$ for $e^{-\lambda x} = a$, i.e. for $x_0 = -\ln a/\lambda$. Since $F''(x)$ changes sign about $x_0 = -\ln a/\lambda$, this is an inflection point. $F(x_0) = F(-\ln a/\lambda) = a/(a+a) = 1/2$. See the graph in Fig. A9.R.16. (d) $\int_{-\infty}^{\infty} f(x)\,dx = \lim_{a\to-\infty}\int_a^0 f(x)dx + \lim_{b\to\infty}\int_0^b f(x)\,dx = \lim_{a\to-\infty}[F(0) - F(a)] + \lim_{b\to\infty}[F(b) - F(0)] = 1$, by (a).

10 Interest Rates and Present Values

10.1

3. We solve $(1 + p/100)^{100} = 100$ for p. Raising each side to $1/100$, $1 + p/100 = \sqrt[100]{100}$, so $p = 100(\sqrt[100]{100} - 1) \approx 100(1.047 - 1) = 4.7$.

5. Use formula (10.1.2). (i) $R = (1 + 0.17/2)^2 - 1 = (1 + 0.085)^2 - 1 = 0.177225$ or 17.72%
For (ii) ansd (iii) see the answers in the text.

10.2

4. If it looses 90% of its value, then $e^{-0.1t^*} = 1/10$, so $-0.1t^* = -\ln 10$, hence $t^* = (\ln 10)/0.1 \approx 23$.

6. With $g(x) = (1 + r/x)^x$ for all $x > 0$, then $\ln g(x) = x\ln(1 + r/x)$. Differentiation gives $g'(x)/g(x) = \ln(1 + r/x) + x(-x/r^2)/(1 + r/x) = \ln(1 + r/x) - (x/r)/(1 + r/x)$, as claimed in the problem. Then see the answer in the text.

10.3

3. (a) We find $f'(t) = 0.05(t + 5)(35 - t)e^{-t}$. Obviously, $f'(t) > 0$ for $t < 35$ and $f'(t) < 0$ for $t > 35$, so $t = 35$ maximizes f (with $f(35) \approx 278$). (b) $f(t) \to 0$ as $t \to \infty$. See the graph in Fig. A10.3.3.

10.4

2. We use formula (10.4.5): (a) $\dfrac{\frac{1}{5}}{1 - \frac{1}{5}} = \dfrac{1}{4}$ (b) $\dfrac{0.1}{1 - 0.1} = \dfrac{0.1}{0.9} = \dfrac{1}{9}$ (c) $\dfrac{517}{1 - 1/1.1} = \dfrac{517 \cdot 1.1}{0.1} = 5687$
(d) $\dfrac{a}{1 - 1/(1 + a)} = 1 + a$ (e) $\dfrac{5}{1 - 3/7} = \dfrac{35}{4}$

6. Let x denote the number of years beyond 1971 that the extractable resources of iron will last. Then $794 + 794\cdot 1.05 + \cdots + 794\cdot(1.05)^x = 249\cdot 10^3$. Using (10.4.3), $794[1 - (1.05)^{x+1}]/(1 - 1.05) = 249\cdot 10^3$ or $(1.05)^{x+1} = 249\cdot 10^3\cdot 0.05/794 \approx 16.68$. Using a calculator, we find $x \approx (\ln 16.68/\ln 1.05) - 1 \approx 56.68$, so the resources will be exhausted part way through the year 2028.

8. (a) The quotient of this infinite series is e^{-rt}, so the sum is $f(t) = \dfrac{P(t)e^{-rt}}{1-e^{-rt}} = \dfrac{P(t)}{e^{rt}-1}$

(b) $f'(t) = \dfrac{P'(t)(e^{rt}-1) - P(t)re^{rt}}{(e^{rt}-1)^2}$, and $t^* > 0$ can only maximize $f(t)$ if $f'(t^*) = 0$, that is, if $P'(t^*)(e^{rt^*}-1) = rP(t^*)e^{rt^*}$, which implies that $\dfrac{P'(t^*)}{P(t^*)} = \dfrac{r}{1-e^{-rt^*}}$.

(c) $\lim\limits_{r\to 0} \dfrac{r}{1-e^{-rt^*}} = \dfrac{\text{“}0\text{”}}{0} = \lim\limits_{r\to 0} \dfrac{1}{t^* e^{-rt^*}} \to \dfrac{1}{t^*}$

10.5

4. Offer (a) is better, because the second offer has present value $4600\dfrac{1-(1.06)^{-5}}{1-(1.06)^{-1}} \approx 20\,540$.

7. This is a geometric series with first term $a = D/(1+r)$ and quotient $k = (1+g)/(1+r)$. It converges iff $k < 1$, i.e. iff $1+g < 1+r$, or $g < r$. The sum is $\dfrac{a}{1-k} = \dfrac{D/(1+r)}{1-(1+g)/(1+r)} = \dfrac{D}{r-g}$.

10.6

4. Schedule (b) has present value $\dfrac{12\,000 \cdot 1.115}{0.115}[1-(1.115)^{-8}] \approx 67\,644.42$.

Schedule (c) has present value $22\,000 + \dfrac{7000}{0.115}[1-(1.115)^{-12}] \approx 66\,384.08$.

Thus schedule (c) is cheapest. When the interest rate becomes 12.5 %, schedules (b) and (c) have present values equal to 65907.61 and 64374.33, respectively.

10.7

5. After dividing all the amounts by \$ 10 000, the equation is $f(s) = s^{20}+s^{19}+\cdots+s^2+s-10 = 0$. Then $f(0) = -10$ and $f(1) = 10$, so by the intermediate value theorem (Theorem 7.10.1), there exists a number s^* such that $f(s^*) = 0$. Here s^* is unique because $f'(s) > 0$. In fact $f(s) = -10+(s-s^{21})/(1-s)$, and $f(s^*) = 0 \iff (s^*)^{10} - 11s^* + 10 = 0$. Then $s^* = 0.928$ is an approximate root, which corresponds to an internal rate of return of about 7.8 %. (See Problem 7.R.26.)

10.8

3. Equilibrium requires $\alpha P_t - \beta = \gamma - \delta P_{t+1}$, or $P_{t+1} = -(\alpha/\delta)P_t + (\beta+\gamma)/\delta$. Using (10.8.4) we obtain the answer in the text.

Review Problems for Chapter 10

3. If you borrow \$ a at the annual interest rate of 11% with interest paid yearly, then the dept after 1 year is $a(1+11/100) = a(1.11)$; if you borrow at annual interest rate 10% with interest paid monthly, your dept after 1 year will be $a(1+10/12\cdot 100)^{12} \approx 1.1047a$, so schedule (ii) is preferable.

6. We use formula (10.4.5): (a) $\dfrac{44}{1-0.56} = 100$ (b) The first term is 20 and the quotient is $1/1.2$, so the sum is $\dfrac{20}{1-1/1.2} = 120$ (c) $\dfrac{3}{1-2/5} = 5$ (d) The first term is $(1/20)^{-2} = 400$ and the quotient is $1/20$, so the sum is $\dfrac{400}{1-1/20)} = 8000/19$

8. (a) See the text. (b) We use formula (10.5.3) on the future value of an annuity. See the text. (c) See the text.

11. (a) $f'(t) = 100e^{\sqrt{t}/2}e^{-rt}\Big[\dfrac{1}{4\sqrt{t}} - r\Big]$. We see that $f'(t) = 0$ for $t^* = 1/16r^2$. Since $f'(t) > 0$ for $t < t^*$ and $f'(t) < 0$ for $t > t^*$, t^* maximizes $f(t)$.
(b) $f'(t) = 200e^{-1/t}e^{-rt}\Big[\dfrac{1}{t^2} - r\Big]$. We see that $f'(t) = 0$ for $t^* = 1/\sqrt{r}$. Since $f'(t) > 0$ for $t < t^*$ and $f'(t) < 0$ for $t > t^*$, t^* maximizes $f(t)$.

12. (a) $F(10) - F(0) = \int_0^{10} (1 + 0.4t)\,dt = \Big|_0^{10} (t + 0.2t^2) = 30$. (Note: the total revenue is $F(10) - F(0)$, not $F(10)$.) (b) See Example 9.5.3.

11 Functions of Many Variables

11.1

6. (a) The denominator must not be 0, so the function is defined for those (x, y) where $y \neq x - 2$.
(b) Must require $2 - (x^2 + y^2) \geq 0$, i.e. $x^2 + y^2 \leq 2$.
(c) Put $a = x^2 + y^2$. We must require $(4 - a)(a - 1) \geq 0$, i.e. $1 \leq a \leq 4$. (Use a sign diagram.)

7. For (a) and (c) see the text. (b) Since $(x - a)^2 \geq 0$ and $(y - b)^2 \geq 0$, it suffices to assume that $x \neq a$ and $b \neq b$, because then we take ln of positive numbers.

11.2

3. (a) and (b) are straightforward. (c) $f(x, y) = (x^2 - 2y^2)^5 = u^5$, where $u = x^2 - 2y^2$. Then $f_1'(x, y) = 5u^4u_1' = 5(x^2 - 2y^2)^4 2x = 10x(x^2 - 2y^2)^4$. In the same way, $f_2'(x, y) = 5u^4u_2' = 5(x^2 - 2y^2)^4(-4y) = -20y(x^2 - 2y^2)^4$. Finally, $f_{12}''(x, y)$ is the derivative of $f_1'(x, y)$ w.r.t. y, keeping x constant, so $f_{12}''(x, y) = (\partial/\partial y)(10x(x^2 - 2y^2)^4) = 10x4(x^2 - 2y^2)^3(-4y) = -160xy(x^2 - 2y^2)^3$.

5. (a)–(c) are easy. (d) $z = x^y = (e^{\ln x})^y = e^{y \ln x} = e^u$ with $u = y \ln x$. Then $z_x' = e^u u_x' = x^y(y/x) = yx^{y-1}$. In the same way, $z_y' = e^u u_y' = x^y \ln x$. Moreover, $z_{xx}'' = (\partial/\partial x)(yx^{y-1}) = y(y - 1)x^{y-2}$. (When differentiating x^{y-1} partially w.r.t. x, y is a constant, so the rule $dx^a/dx = ax^{a-1}$ applies.) $z_{yy}'' = (\partial/\partial y)(x^y \ln x) = x^y(\ln x)^2$. Finally, $z_{xy}'' = (\partial/\partial y)(yx^{y-1}) = x^{y-1} + yx^{y-1}\ln x$. (Note that if $w = x^{y-1} = x^v$, with $v = y - 1$, then $w_y' = x^v \ln x \cdot 1 = x^{y-1}\ln x$. Or: $w = x^{y-1} = (1/x)x^y$ etc.)

11.3

8. (a) The point (2, 3) lies on the level curve $z = 8$, so $f(2, 3) = 8$. The points $(x, 3)$ are those on the line $y = 3$ parallel to the x-axis. This line intersects the level curve $z = 8$ when $x = 2$ and $x = 5$.
(b) See the text. (c) If at A you look in the direction of the positive x-axis, you meet level sets whose values increase, so $f_1'(x, y) > 0$. A rough estimate of $f_1'(x, y)$ at A is 2, because if you go one unit in the positive x-axis direction from A, then $f(x, y)$ increases from 8 to 10.

9. (a) It might help to regard the figure as a map of a mountain. At P the terrain is raising in the direction of the positive x-axis, so $f_x'(P) > 0$. The terrain is sloping downwards in the direction of the positive y-axis so $f_y'(P) < 0$. (b) (i) The line $x = 1$ has no point in common with any of the given level curves.

(ii) The line $y = 2$ intersects the level curve $z = 2$ at $x = 2$ and $x = 6$ (approximately).
(c) If you start at the point $(6, 0)$ and move up along the line $2x + 3y = 12$, you first meet the level curve $z = f(x, y) = 1$. Moving further you meet level curves with higher z-values. The level curve with the highest z-value you meet is $z = 3$, where the straight just touches the level curve.

10. $F(1,0) = F(0,0) + \int_0^1 F_1'(x,0)\,dx \geq \int_0^1 2\,dx = 2$, $F(2,0) = F(1,0) + \int_1^2 F_1'(x,0)\,dx \geq F(1,0) + 2$,
$F(0,1) = F(0,0) + \int_0^1 F_2'(0,y)\,dy \leq 1$, $F(1,1) = F(0,1) + \int_0^1 F_1'(x,1)\,dx \geq F(0,1) + 2$,
$F(1,1) = F(1,0) + \int_0^1 F_2'(1,y)\,dy \leq F(1,0) + 1$.

11.6

2. (a)–(d) are routine. (e) $f(x,y,z) = (x^2+y^3+z^4)^6 = u^6$, with $u = x^2+y^3+z^4$. Then $f_1' = 6u^5u_1' = 6(x^2+y^3+z^4)^5 2x = 12x(x^2+y^3+z^4)^5$, $f_2' = 6u^5u_2' = 6(x^2+y^3+z^4)^5 3y^2 = 18y^2(x^2+y^3+z^4)^5$, $f_3' = 6u^5u_3' = 6(x^2+y^3+z^4)^5 4z^3 = 24z^3(x^2+y^3+z^4)^5$ (f) $f(x,y,z) = e^{xyz} = e^u$, with $u = xyz$, gives $f_1' = e^u u_1' = e^{xyz}yz$. Similarly, $f_2' = e^u u_2' = e^{xyz}xz$, and $f_3' = e^u u_3' = e^{xyz}xy$

5. When r and w are constants, so is $(1/r + 1/w)$, and thus $\partial\pi/\partial p = \frac{1}{2}p(1/r + 1/w)$.

10. From $f = x^{y^z}$ we get $(*)$ $\ln f = y^z \ln x$. Differentiating $(*)$ w.r.t x yields $f_x'/f = y^z/x$, and so $f_x' = fy^z/x = x^{y^z}y^z/x = y^z x^{y^z-1}$. Differentiating $(*)$ w.r.t y yields $f_y'/f = zy^{z-1}\ln x$, and so $f_y' = zy^{z-1}(\ln x)x^{y^z}$. Differentiating $(*)$ w.r.t z yields $f_z'/f = y^z(\ln y)(\ln x)$, and so $f_z' = y^z(\ln x)(\ln y)x^{y^z}$.

11.7

2. (a) $Y_K' = aAK^{a-1}$ and $Y_K' = aBL^{a-1}$, so $KY_K' + LY_L' = aAK^a + aBL^a = a(AK^a + BL^a) = aY$
(b) $KY_K' + LY_L' = KaAK^{a-1}L^b + LAK^a bL^{b-1} = aAK^aL^b + bAK^aL^a = (a+b)AK^aL^b = (a+b)Y$
(c) $Y_K' = \dfrac{2aKL^5 - bK^4L^2}{(aL^3+bK^3)^2}$ and $Y_L' = \dfrac{2bK^5L - aK^2L^4}{(aL^3+bK^3)^2}$, so

$$KY_K' + LY_L' = \frac{2aK^2L^5 - bK^5L^2 + 2bK^5L^2 - aK^2L^5}{(aL^3+bK^3)^2} = \frac{K^2L^2(aL^3+bK^3)}{(aL^3+bK^3)^2} = \frac{K^2L^2}{aL^3+bK^3} = Y.$$

(According to Section 12.6 these functions are homogeneous of degrees a, $a+b$, and 1, respectively, so the results we obtained are immediate consequences of Euler's Theorem, (12.6.2).)

7. $Y_K' = (-m/\rho)a(-\rho)K^{-\rho-1}Ae^{\lambda t}\big[aK^{-\rho} + bL^{-\rho}\big]^{-(m/\rho)-1} = maK^{-\rho-1}Ae^{\lambda t}\big[aK^{-\rho} + bL^{-\rho}\big]^{-(m/\rho)-1}$,
$Y_L' = (-m/\rho)b(-\rho)L^{-\rho-1}Ae^{\lambda t}\big[aK^{-\rho} + bL^{-\rho}\big]^{-(m/\rho)-1} = mbL^{-\rho-1}Ae^{\lambda t}\big[aK^{-\rho} + bL^{-\rho}\big]^{-(m/\rho)-1}$.
Thus, $KY_K' + LY_L' = m(aK^{-\rho} + bL^{-\rho})Ae^{\lambda t}\big[aK^{-\rho} + bL^{-\rho}\big]^{-(m/\rho)-1} = mY$. (This function is homogeneous of degree m, so the result is an immediate consequences of Euler's Theorem, (12.6.2).)

11.8

4. $\dfrac{\partial}{\partial m}\Big(\dfrac{pD}{m}\Big) = p\dfrac{mD_m' - D}{m^2} = \dfrac{p}{m^2}(mD_m' - D) = \dfrac{pD}{m^2}[\mathrm{El}_m D - 1] > 0$ iff $\mathrm{El}_m D > 1$,
so pD/m increases with m if $\mathrm{El}_m D > 1$. (Using the formulas in Problem 7.7.9, the result also follows from the fact that $\mathrm{El}_m(pD/m) = \mathrm{El}_m p + \mathrm{El}_m D - \mathrm{El}_m m = \mathrm{El}_m D - 1$.)

Review Problems for Chapter 11

7. (a) $z = (x^2y^4+2)^5 = u^5$, with $u = x^2y^4+2$, so $\dfrac{\partial z}{\partial x} = 5u^4\dfrac{\partial u}{\partial x} = 5(x^2y^4+2)^4 2xy^4 = 10xy^4(x^2y^4+2)^4$
For the rest see the text.

11. (b) We want to find all (x, y) that satisfy both equations (i) $4x^3 - 8xy = 0$ and (ii) $4y - 4x^2 + 4 = 0$. From (i), $4x(x^2 - 2y) = 0$, which means that $x = 0$, or $x^2 = 2y$. For $x = 0$, (ii) yields $y = -1$, so $(x, y) = (0, -1)$ is one solution. With $x^2 = 2y$, (ii) reduces to $4y - 8y + 4 = 0$, or $y = 1$. But then $x^2 = 2$, such that $x = \pm\sqrt{2}$. So the two additional solutions are $(x, y) = (\pm\sqrt{2}, 1)$.

12 Tools for Comparative Statics

12.1

5. We look at (c) and (d). (c): If $z = F(x, y) = xy$ with $x = f(t)$ and $y = g(t)$, then $F_1'(x, y) = y$, $F_2'(x, y) = x$, $dx/dt = f'(t)$, and $dy/dt = g'(t)$, so formula (12.1.1) gives $dz/dt = F_1'(x, y)(dx/dt) + F_2'(x, y)(dy/dt) = yf'(t) + xg'(t) = g(t)f'(t) + f(t)g'(t)$.
(d): If $z = F(x, y) = \dfrac{x}{y}$ with $x = f(t)$ and $y = g(t)$, then $F_1'(x, y) = \dfrac{1}{y}$, $F_2'(x, y) = -\dfrac{x}{y^2}$, $\dfrac{dx}{dt} = f'(t)$, and $\dfrac{dy}{dt} = g'(t)$, so formula (12.1.1) gives $\dfrac{dz}{dt} = F_1'(x, y)\dfrac{dx}{dt} + F_2'(x, y)\dfrac{dy}{dt} = \dfrac{1}{y}f'(t) - \dfrac{x}{y^2}g'(t) = \dfrac{yf'(t) - xg'(t)}{y^2} = \dfrac{g(t)f'(t) - f(t)g'(t)}{(g(t))^2}$.

6. Let $U(x) = u(x, h(x)) = \ln[x^\alpha + (ax^4 + b)^{\alpha/3}] - \frac{\alpha}{3}\ln(ax^4 + b)$.
Then $U'(x) = \dfrac{\alpha x^{\alpha-1}(3b - ax^4)}{3[x^\alpha + (ax^4 + b)^{\alpha/3}](ax^4 + b)}$. So $U'(x^*) = 0$ at $x^* = \sqrt[4]{3b/a}$, whereas $U'(x) > 0$ for $x < x^*$ and $U'(x) < 0$ for $x > x^*$. Hence x^* maximizes U.

7. Differentiating (12.1.1) w.r.t. t yields, $d^2z/dt^2 = (d/dt)[F_1'(x, y)\,dx/dt] + (d/dt)[F_2'(x, y)\,dy/dt]$. Here $(d/dt)[F_1'(x, y)\,dx/dt] = [F_{11}''(x, y)\,dx/dt + F_{12}''(x, y)\,dy/dt]dx/dt + F_1'(x, y)\,d^2x/dt^2$, $(d/dt)[F_2'(x, y)\,dy/dt] = [F_{21}''(x, y)\,dx/dt + F_{22}''(x, y)\,dy/dt]\,dy/dt + F_2'(x, y)\,d^2y/dt^2$. Assuming $F_{12}'' = F_{21}''$, the conclusion follows.

12.2

2. (a) Let $z = F(x, y) = xy^2$ with $x = t + s^2$ and $y = t^2s$. Then $F_1'(x, y) = y^2$, $F_2'(x, y) = 2xy$, $\partial x/\partial t = 1$, and $\partial y/\partial t = 2ts$. Then (12.2.1) gives $\partial z/\partial t = F_1'(x, y)(\partial x/\partial t) + F_2'(x, y)(\partial y/\partial t) = y^2 + 2xy2ts = (t^2s)^2 + 2(t + s^2)t^2s2ts = t^3s^2(5t + 4s^2)$. $\partial z/\partial s$ is found in the same way.
(b) $\dfrac{\partial z}{\partial t} = F_1'(x, y)\dfrac{\partial x}{\partial t} + F_2'(x, y)\dfrac{\partial y}{\partial t} = \dfrac{2y}{(x + y)^2}e^{t+s} + \dfrac{-2sx}{(x + y)^2}e^{ts}$, etc.

3. It is important that you can do these problems, because in economic applications, functions are frequently not completely specified. (a) $z_r' = F_u'u_r' + F_v'v_r' + F_w'w_r' = F_u'2r + F_v' \cdot 0 + F_w'(1/r) = 2rF_u' + (1/r)F_w'$. Check that you get the same answers as in the text.

7. Use the formulas in (12.2.1) and see the text. Only the notation is different.

8. (a) Let $u = \ln v$, where $v = x^3 + y^3 + z^3 - 3xyz$. Then $\partial u/\partial x = (1/v)(\partial v/\partial x) = (3x^2 - 3yz)/v$. Similarly, $\partial u/\partial y = (3y^2 - 3xz)/v$, and $\partial u/\partial z = (3z^2 - 3xy)/v$. Hence,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{1}{v}(3x^3 - 3xyz) + \frac{1}{v}(3y^3 - 3xyz) + \frac{1}{v}(3z^3 - 3xyz) = \frac{3v}{v} = 3$$

which proves (i). Equation (ii) is then proved by elementary algebra.
(b) Note that f is here a function of *one* variable. With $z = f(u)$ where $u = x^2 y$, we get $\partial z/\partial x = f'(u)u'_x = 2xyf'(x^2y)$. Likewise, $\partial z/\partial y = x^2 f'(x^2 y)$, so $x\partial z/\partial x = 2x^2 yf'(x^2y) = 2y\partial z/\partial y$.

12.3

2. (a) See the text. (b) Put $F(x, y) = x - y + 3xy$. Then $F'_1 = 1 + 3y$, $F'_2 = -1 + 3x$, $F''_{11} = 0$, $F''_{12} = 3$, $F''_{22} = 0$. In particular, $y' = -F'_1/F'_2 = -(1 + 3y)/(-1 + 3x)$. Moreover, using equation (12.3.3),
$$y'' = -\frac{1}{(F'_2)^3}\big[F''_{11}(F'_2)^2 - 2F''_{12}F'_1F'_2 + F''_{22}(F'_1)^2\big] = \frac{6(1+3y)(-1+3x)}{(-1+3x)^3}) = \frac{6(1+3y)}{(-1+3x)^2}.$$
(c) Put $F(x, y) = y^5 - x^6$. Then $F'_1 = -6x^5$, $F'_2 = 5y^4$, $F''_{11} = -30x^4$, $F''_{12} = 0$, $F''_{22} = 20y^3$, so $y' = -F'_1/F'_2 = -(-6x^5/5y^4) = 6x^5/5y^4$. Moreover, using equation (12.3.3),
$$y'' = -\frac{1}{(5y^4)^3}\big[(-30x^4)(5y^4)^2 + 20y^3(-6x^5)^2\big] = \frac{6x^4}{y^4} - \frac{144x^{10}}{25y^9}.$$

3. (a) With $F(x, y) = 2x^2 + xy + y^2$, $y' = -F'_1/F'_2 = -(4x + y)/(x + 2y) = -4$ at $(2, 0)$. Moreover, $y'' = -(28x^2 + 14y^2 + 14xy)/(x + 2y)^3 = -14$ at $(2, 0)$. The tangent has the equation $y = -4x + 8$.
(b) $y' = 0$ requires $y = -4x$. Inserting this into the original equation gives the two points.

4. If we define $F(x, y) = 3x^2 - 3xy^2 + y^3 + 3y^2$, the given equation is $F(x, y) = 4$. Now, $F'_1(x, y) = 6x - 3y^2$ and $F'_2(x, y) = -6xy + 3y^2 + 6y$, so according to (12.3.1), $y' = -(6x - 3y^2)/(-6xy + 3y^2 + 6y)$.

12.4

1. (a) and (b) are easy. (c) Defining $F(x, y, z) = e^{xyz} - 3xyz$, the given equation is $F(x, y, z) = 0$. Now, $F'_x(x, y, z) = yze^{xyz} - 3yz$, $F'_z(x, y, z) = xye^{xyz} - 3xy$, so (12.4.1) gives, $z'_x = -F'_x/F'_z = -(yze^{xyz} - 3yz)/(xye^{xyz} - 3xy) = -yz(e^{xyz} - 3)/xy(e^{xyz} - 3) = -z/x$. (Actually, the equation $e^{xyz} = 3xyz$ has two constant solutions. From $xyz = c$ we find z'_x much easier.)

3. (a) Equation $(*)$ is here $P/2\sqrt{L^*} = w$. Solve for L^*. (b) The first-order condition is now
$$Pf'(L^*) - C'_L(L^*, w) = 0 \tag{$*$}$$
Differentiate $(*)$ partially w.r.t. P keeping in mind that L^* depends on P. To find the partial derivative of $Pf'(L^*)$ w.r.t. P use the product rule to get $1 \cdot f'(L^*) + Pf''(L^*)(\partial L^*/\partial P)$. The partial derivative of $C'_L(L^*, w)$ w.r.t. P is $C''_{LL}(L^*, w)(\partial L^*/\partial P)$. So, all in all, $f'(L^*) + Pf''(L^*)(\partial L^*/\partial P) - C''_{LL}(L^*, w)(\partial L^*/\partial P) = 0$. Then solve for $\partial L^*/\partial P$.

Differentiating $(*)$ w.r.t. w gives, $Pf''(L^*)(\partial L^*/\partial w) - C''_{LL}(L^*, w)(\partial L^*/\partial w) - C''_{Lw}(L^*, w) = 0$. Then solve for $\partial L^*/\partial w$.

6. (a) $F'_1(x, y) = e^{y-3} + y^2$ and $F'_2(x, y) = xe^{y-3} + 2xy - 2$. Hence, the slope of the tangent to the level curve $F(x, y) = 4$ at the point $(1, 3)$ is $y' = -F'_1(1, 3)/F'_2(1, 3) = -10/5 = -2$.
(b) Taking the logarithm of both sides, we get $(1 + c\ln y)\ln y = \ln A + \alpha\ln K + \beta\ln L$. Differentiation with respect to K gives $\dfrac{c}{y}\dfrac{\partial y}{\partial K}\ln y + (1 + c\ln y)\dfrac{1}{y}\dfrac{\partial y}{\partial K} = \dfrac{\alpha}{K}$. Solving for $\partial y/\partial K$ yields the given answer. $\partial y/\partial K$ is found in the same way.

12.5

3. With $F(K,L) = AK^aL^b$, $F'_K = aF/K$, $F'_L = bF/L$, $F''_{KK} = a(a-1)F/K^2$, $F''_{KL} = abF/KL$, and $F''_{LL} = b(b-1)F/L^2$. Also, $-F'_K F'_L(KF'_K + LF'_L) = -(aF/K)(bF/L)(a+b)F = -ab(a+b)F^3/KL$. Moreover, $KL\big[(F'_L)^2 F''_{KK} - 2F'_K F'_2 F''_{KL} + (F'_K)^2 F''_{LL}\big] = -ab(a+b)F^3/KL$. It follows that $\sigma_{KL} = 1$.

12.6

3. (2): $xf'_1(x,y) + yf'_2(x,y) = x(y^2+3x^2) + y2xy = 3(x^3+xy^2) = 3f(x,y)$
(3): It is easy to see that $f'_1(x,y) = y^2+3x^2$ and $f'_2(x,y) = 2xy$ are homogeneous of degree 2.
(4): $f(x,y) = x^3+xy^2 = x^3[1+(y/x)^2] = y^3[(x/y)^3 + x/y]$
(5): $x^2 f''_{11} + 2xyf''_{12} + y^2 f''_{22} = x^2(6x) + 2xy(2y) + y^2(2x) = 6x^3 + 4xy^2 + 2xy^2 = 3\cdot 2f(x,y)$

4. Using the results in Example 11.2.1(b), $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = \dfrac{xy^3 - x^3y + x^3y - xy^3}{(x^2+y^2)^2} = 0 = 0\cdot f$, so f is homogeneous of degree 0.

8. From (∗), with $k = 1$, we get $f''_{11} = (-y/x)f''_{12}$ and $f''_{22} = (-x/y)f''_{21}$. With $f''_{12} = f''_{21}$ we get $f''_{11}f''_{22} - (f''_{12})^2 = (-y/x)f''_{12}(-x/y)f''_{12} - (f''_{12})^2 = 0$.

12.7

1. (a) and (f) are easy. In (b) you can use Euler's theorem, as in (e) below.
(c) $h(tx,ty,tz) = \dfrac{\sqrt{tx}+\sqrt{ty}+\sqrt{tz}}{tx+ty+tz} = \dfrac{\sqrt{t}\,(\sqrt{x}+\sqrt{y}+\sqrt{z})}{t(x+y+z)} = t^{-1/2}h(x,y,z)$ for all $t > 0$, so h is homogeneous of degree $-1/2$. (d) $G(tx,ty) = \sqrt{txty}\,\ln\dfrac{(tx)^2+(ty)^2}{txty} = t\sqrt{xy}\,\ln\dfrac{t^2(x^2+y^2)}{t^2xy} = tG(x,y)$ for all $t > 0$, so G is homogeneous of degree 1. (e) $xH'_x + yH'_y = x(1/x) + y(1/y) = 2$. Since 2 is not equal to $k(\ln x + \ln y)$ for any constant k, by Euler's theorem, H is not homogeneous of any degree.

2. (a) $f(tx_1,tx_2,tx_3) = \dfrac{(tx_1tx_2tx_3)^2}{(tx_1)^4+(tx_2)^4+(tx_3)^4}\left(\dfrac{1}{tx_1}+\dfrac{1}{tx_2}+\dfrac{1}{tx_3}\right) =$
$\dfrac{t^6(x_1x_2x_3)^2}{t^4(x_1^4+x_2^4+x_3^4)}\left(\dfrac{1}{t}\right)\left(\dfrac{1}{x_1}+\dfrac{1}{x_2}+\dfrac{1}{x_3}\right) = tf(x_1,x_2,x_3)$, so f is homogeneous of degree 1.
(b) $x(tv_1,tv_2,\dots,tv_n) = A\big(\delta_1(tv_1)^{-\varrho} + \delta_2(tv_2)^{-\varrho} + \cdots + \delta_n(tv_n)^{-\varrho}\big)^{-\mu/\varrho} =$
$A\big(t^{-\varrho}(\delta_1v_1^{-\varrho} + \delta_2v_2^{-\varrho} + \cdots + \delta_nv_n^{-\varrho})\big)^{-\mu/\varrho} = (t^{-\varrho})^{-\mu/\varrho}A\big(\delta_1v_1^{-\varrho} + \delta_2v_2^{-\varrho} + \cdots + \delta_nv_n^{-\varrho}\big)^{-\mu/\varrho} =$
$t^{\mu}A\big(\delta_1v_1^{-\varrho} + \delta_2v_2^{-\varrho} + \cdots + \delta_nv_n^{-\varrho}\big)^{-\mu/\varrho} = t^{\mu}x(x_1,x_2,x_3)$, so x is homogeneous of degree μ.

12.8

1. In both (a) and (b) we use the approximation $f(x,y) \approx f(0,0) + f'_1(0,0)x + f'_2(0,0)y$.
(a) For $f(x,y) = \sqrt{1+x+y}$, $f(0,0) = 1$, and $f'_1(x,y) = f'_2(x,y) = \dfrac{1}{2\sqrt{1+x+y}}$, so $f'_1(0,0) = f'_2(0,0) = 1/2$, and the linear approximation to $f(x,y)$ about $(0,0)$ is $f(x,y) \approx 1 + \frac{1}{2}x + \frac{1}{2}y$.
(b) For $f(x,y) = e^x\ln(1+y)$, $f'_1(x,y) = e^x\ln(1+y)$ and $f'_2(x,y) = \dfrac{e^x}{1+y}$. Here, $f(0,0) = 0$, $f'_1(0,0) = e^0\ln 1 = 0$ and $f'_2(0,0) = 1$. That yields $f(x,y) = e^x\ln(1+y) \approx 0 + 0\cdot x + 1\cdot y = y$.

3. $\dfrac{\partial g^*}{\partial \mu} = \dfrac{1}{1-\beta}\big[(1+\mu)(1+\varepsilon)^\alpha\big]^{1/(1-\beta)-1}(1+\varepsilon)^\alpha = \dfrac{1}{1-\beta}\big[(1+\mu)(1+\varepsilon)^\alpha\big]^{\beta/(1-\beta)}(1+\varepsilon)^\alpha$,
$\dfrac{\partial g^*}{\partial \varepsilon} = \dfrac{1}{1-\beta}\big[(1+\mu)(1+\varepsilon)^\alpha\big]^{\beta/(1-\beta)}(1+\mu)\alpha(1+\varepsilon)^{\alpha-1}$. See the text.

7. We use formula (12.8.3). (a) Here, $\partial z/\partial x = 2x$ and $\partial z/\partial y = 2y$. At (1, 2, 5), we get $\partial z/\partial y = 2$ and $\partial z/\partial x = 4$, so the tangent plane has the equation $z - 5 = 2(x-1) + 4(y-2) \iff z = 2x + 4y - 5$. (b) From $z = (y-x^2)(y-2x^2) = y^2 - 3x^2y + 2x^4$ we get $\partial z/\partial x = -6xy + 8x^3$ and $\partial z/\partial y = 2y - 3x^2$. Thus, at (1, 3, 2) we have $\partial z/\partial x = -10$ and $\partial z/\partial y = 3$. The tangent plane is given by the equation $z - 2 = -10(x-1) + 3(y-3) \iff z = -10x + 3y + 3$.

12.9

2. We can either use the definition of the differential, (12.9.1), or the rules for differentials as we do here. (a) $dz = d(x^3) + d(y^3) = 3x^2dx + 3y^2dy$ (b) $dz = (dx)e^{y^2} + x(de^{y^2})$. Here $de^{y^2} = e^{y^2}dy^2 = e^{y^2}2ydy$, so $dz = e^{y^2}dx + 2xye^{y^2}dy = e^{y^2}(dx + 2xydy)$.
(c) $dz = d\ln u$, where $u = x^2 - y^2$. Then $dz = \dfrac{1}{u}du = \dfrac{2xdx - 2ydy}{x^2 - y^2}$.

5. $d(Ue^U) = d(x\sqrt{y})$, and so $e^U dU + Ue^U dU = dx\sqrt{y} + (x/2\sqrt{y})dy$. Solving for dU yields the answer.

12.11

3. Since we are asked to find the partials of y_1 and y_2 w.r.t. x_1 only, we might as well differentiate the system partially w.r.t. x_1:

(i) $3 - \dfrac{\partial y_1}{\partial x_1} - 9y_2^2\dfrac{\partial y_2}{\partial x_1} = 0$ (ii) $3x_1^2 + 6y_1^2\dfrac{\partial y_1}{\partial x_1} - \dfrac{\partial y_2}{\partial x_1} = 0$

Solve for the partials and see the text.
(An alternative, in particular if one needs all the partials, is to use total differentiation:

(i) $3\,dx_1 + 2x_2\,dx_2 - dy_1 - 9y_2^2\,dy_2 = 0$, (ii) $3x_1^2\,dx_1 - 2dx_2 + 6y_1^2\,dy_1 - dy_2 = 0$

Letting $dx_2 = 0$ and solving for dy_1 and dy_2 leads to $dy_1 = A dx_1$ and $dy_2 = B dx_1$, where $A = \partial y_1/\partial x_1$ and $B = \partial y_2/\partial x_1$.)

4. Differentiation with respect to M gives, (i) $I'(r)r'_M = S'(Y)Y'_M$, (ii) $aY'_M + L'(r)r'_M = 1$. (Remember that Y and r are functions of the independent variables a and M.) Writing this as a linear equation system on standard form, we get

$$\begin{aligned} -S'(Y)Y'_M + I'(r)r'_M &= 0 \\ aY'_M + L'(r)r'_M &= 1 \end{aligned}$$

Cramer's rule (or use ordinary elimination) gives

$$Y'_M = \frac{\begin{vmatrix} 0 & I'(r) \\ 1 & L'(r) \end{vmatrix}}{\begin{vmatrix} -S'(Y) & I'(r) \\ a & L'(r) \end{vmatrix}} = \frac{I'(r)}{S'(Y)L'(r) + aI'(r)} \quad \text{and} \quad r'_M = \frac{S'(Y)}{S'(Y)L'(r) + aI'(r)}$$

5. Differentiation w.r.t. x yields, $y + u'_x v + u v'_x = 0$, $u + x u'_x + y v'_x = 0$. Solving this system for u'_x and v'_x we get

$$u'_x = \frac{u^2 - y^2}{yv - xu} = \frac{u^2 - y^2}{2yv}, \quad v'_x = \frac{xy - uv}{yv - xu} = \frac{2xy - 1}{2yv}$$

where we substituted $xu = -yv$ and $uv = 1 - xy$. Differentiating u'_x w.r.t. x finally yields

$$u''_{xx} = \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} u'_x = \frac{2uu'_x 2yv - (u^2 - y^2) 2y v'_x}{4y^2 v^2} = \frac{(u^2 - y^2)(4uv - 1)}{4y^2 v^3}$$

9. (a) Differentiation yields $2uv\,du + u^2 dv - du = 3x^2 dx + 6y^2 dy$, and $e^{ux}(u\,dx + x\,du) = v\,dy + y\,dv$. At P these equations become $3du + 4dv = 6dy$ and $dv = 2dx - dy$. Hence $du = 2dy - (4/3)dv = -(8/3)\,dx + (10/3)\,dy$. So $\partial u/\partial y = 10/3$, $\partial v/\partial x = 2$. (b) See the text.

Review Problems for Chapter 12

4. $X = Ng(u)$, where $u = \varphi(N)/N$. Then $du/dN = [\varphi'(N)N - \varphi(N)]/N^2 = (1/N)(\varphi'(N) - u)$, and by the product rule and the chain rule,

$$\frac{dX}{dN} = g(u) + Ng'(u)\frac{du}{dN} = g(u) + g'(u)(\varphi'(N) - u), \quad u = \frac{\varphi(N)}{N}$$

Differentiating $g(u) + g'(u)(\varphi'(N) - u)$ w.r.t. N gives

$$\begin{aligned}\frac{d^2X}{dN^2} &= g'(u)\frac{du}{dN} + g''(u)\frac{du}{dN}(\varphi'(N) - u) + g'(u)(\varphi''(N) - \frac{du}{dN}) \\ &= \frac{1}{N} g''\big(\varphi(N)/N\big)\big[\varphi'(N) - \varphi(N)/N\big]^2 + g'\big(\varphi(N)/N\big)\varphi''(N)\end{aligned}$$

5. (a) Take the natural logarithm, $\ln E = \ln A - a \ln p + b \ln m$, and then differentiate. (b) See the text.

11. $\mathrm{El}_x(y^2 e^x e^{1/y}) = \mathrm{El}_x\, y^2 + \mathrm{El}_x\, e^x + \mathrm{El}_x\, e^{1/y} = 0$. Here $\mathrm{El}_x\, y^2 = 2\,\mathrm{El}_x\, y$ and $\mathrm{El}_x\, e^x = x$. Moreover, $\mathrm{El}_x\, e^{1/y} = \mathrm{El}_x\, e^u$, where $u = 1/y$, so $\mathrm{El}_x\, e^u = u\,\mathrm{El}_x(1/y) = (1/y)(\mathrm{El}_x 1 - \mathrm{El}_x\, y) = -(1/y)\,\mathrm{El}_x\, y$. All in all, $2\,\mathrm{El}_x\, y + x - (1/y)\,\mathrm{El}_x\, y = 0$, so $\mathrm{El}_x\, y = xy/(1 - 2y)$. (We used the rules for elasticities in Problem 7.7.9. If you are not comfortable with these rules, find y' by implicit differentiation and then use $\mathrm{El}_x\, y = (x/y)y'$.

16. (a) Differentiating and then gathering all terms in dp and dL on the left-hand side, yields

$$\text{(i) } F'(L)\,dp + pF''(L)\,dL = dw \qquad \text{(ii) } F(L)\,dp + (pF'(L) - w)\,dL = L\,dw + dB$$

Since we know that $pF'(L) = w$, (ii) implies that $dp = (Ldw + dB)/F(L)$. Substituting this into (i) and solving for dL, we obtain $dL = [(F(L) - LF'(L))dw - F'(L)dB]/pF(L)F''(L)$. It follows that

$$\frac{\partial p}{\partial w} = \frac{L}{F(L)}, \quad \frac{\partial p}{\partial B} = \frac{1}{F(L)}, \quad \frac{\partial L}{\partial w} = \frac{F(L) - LF'(L)}{pF(L)F''(L)}, \quad \frac{\partial L}{\partial B} = -\frac{F'(L)}{pF(L)F''(L)}$$

(b) We know that $p > 0$, $F'(L) > 0$, and $F''(L) < 0$. Also, $F(L) = (wL + B)/p > 0$. Hence, it is clear that $\partial p/\partial w > 0$, $\partial p/\partial B > 0$, and $\partial L/\partial B > 0$.

To find the sign of $\partial L/\partial w$, we need the sign of $F(L) - LF'(L)$. From the equations in the model, we get $F'(L) = w/p$ and $F(L) = (wL + B)/p$, so $F(L) - LF'(L) = B/p > 0$. Therefore $\partial L/\partial w < 0$.

13 Multivariable Optimization

13.1

3. $F'_K = -2(K-3) - (L-6)$ and $F'_L = -4(L-6) - (K-3)$, so the first-order conditions yield $-2(K-3) - (L-6) = 0.65$, $-4(L-6) - (K-3) = 1.2$. The solution is $(K, L) = (2.8, 5.75)$.

4. (b) First-order conditions: $P'_x = -2x + 22 = 0$, $P'_y = -2y + 18 = 0$. It follows that $x = 11$ and $y = 9$.

13.2

3. Note that $U = (108 - 3y - 4z)yz$. Then $\partial U/\partial y = 108z - 6yz - 4z^2 = 0$ and $\partial U/\partial z = 108y - 3y^2 - 8yz = 0$. Because y and z are assumed to be positive, these two equations reduce to $6y + 4z = 108$ and $3y + 8z = 108$, with solution $y = 12$ and $z = 9$. Theorem 13.2.1 cannot be used directly to prove optimality. However, it can be applied to the equivalent problem of maximizing $\ln z$. See Theorem 13.6.3.)

7. Solve the constraint for z: $z = 4x + 2y - 5$. Then minimize $P(x, y) = x^2 + y^2 + (4x + 2y - 5)^2$ w.r.t. x and y. The first-order conditions are: $P'_1 = 34x + 16y - 40 = 0$, $P'_2 = 16x + 10y - 20 = 0$, with solution $x = 20/21$, $y = 10/21$. Since $P''_{11} = 34$, $P''_{12} = 16$, and $P''_{22} = 10$, we see that the second-order conditions for minimum are satisfied.

13.3

3. (a) $V'_t(t, x) = f'_t(t, x)e^{-rt} - rf(t, x)e^{-rt} = 0$, $V'_x(t, x) = f'_x(t, x)e^{-rt} - 1 = 0$, so at the optimum, $f'_t(t^*, x^*) = rf(t^*, x^*)$ and $f'_x(t^*, x^*) = e^{rt^*}$. (b) See the text and (c).
(c) $V(t, x) = g(t)h(x)e^{-rt} - x$, so $V'_t = h(x)(g'(t) - rg(t))e^{-rt}$, $V'_x = g(t)h'(x)e^{-rt} - 1$. Moreover, $V''_{tt} = h(x)(g''(t) - 2rg'(t) + r^2 g(t))e^{-rt}$, $V''_{tx} = h'(x)(g'(t) - rg(t))e^{-rt}$, and $V''_{xx} = g(t)h''(x)e^{-rt}$. At (t^*, x^*), $V''_{tx} = 0$, $V''_{xx} < 0$, and $V''_{tt} = h(x^*)[g''(t^*) - 2rg'(t^*) + r^2 g(t^*)]e^{-rt^*}$. Because $g'(t^*) = rg(t^*)$, we obtain $V''_{tt} = h(x^*)[g''(t^*) - r^2 g(t^*)]e^{-rt^*} < 0$. Thus (t^*, x^*) is a local maximum point.
(d) The first-order conditions in (b) reduce to $e^{\sqrt{t^*}}/2\sqrt{t^*} = re^{\sqrt{t^*}}$, so $t^* = 1/4r^2$, and $1/(x^* + 1) = e^{1/4r}/e^{1/2r}$, or $x^* = e^{1/4r} - 1$. The two conditions in (c) are satisfied. (Note that in part (c) of this problem in the text, the second condition should be $h''(x^*) < 0$.) Obviously, $h''(x^*) = -(1 + x^*)^{-2} < 0$. Moreover, $g''(t^*) = \dfrac{1}{4t^*\sqrt{t^*}} e^{\sqrt{t^*}}(\sqrt{t^*} - 1) = r^2(1 - 2r)e^{1/2r} < r^2 e^{1/2r}$, which is true when $r > 0$.

5. (a) We need to have $1 + x^2 y > 0$. When $x = 0$, $f(0, y) = 0$. For $x \neq 0$, $1 + x^2 y > 0 \iff y > -1/x^2$. (The figure in the text shows a part of the graph of f. Note that $f = 0$ on the x-axis and on the y-axis.)
(b) See the text. (c) $f''_{11}(x, y) = \dfrac{2y - 2x^2y^2}{(1 + x^2y)^2}$, $f''_{12}(x, y) = \dfrac{2x}{(1 + x^2y)^2}$, and $f''_{22}(x, y) = \dfrac{-x^4}{(1 + x^2y)^2}$. The second-order derivatives at all points of the form $(0, b)$ are $f''_{11}(0, b) = 2b$, $f''_{12}(0, b) = 0$, and $f''_{22}(0, b) = 0$. Since $f''_{11}f''_{22} - (f''_{12})^2 = 0$ at all the stationary points, the second-derivative test tells us nothing about the stationary points. See the text.

13.4

2. (a) See the text. (b) The new profit function is $\hat{\pi} = -bp^2 - dp^2 + (a+\beta b)p + (c+\beta d)p - \alpha - \beta(a+c)$ and the price which maximizes profits is easily seen to be $\hat{p} = \dfrac{a+c+\beta(b+d)}{2(b+d)}$.

(c) If $\beta = 0$, then $p^* = \dfrac{a}{2b}$, $q^* = \dfrac{c}{2d}$, and $\pi(p^*, q^*) = \dfrac{a^2}{4b} + \dfrac{c^2}{4d} - \alpha$. Moreover, $\hat{p} = \dfrac{a+c}{2(b+d)}$ with $\hat{\pi}(\hat{p}) = \dfrac{(a+c)^2}{4(b+d)} - \alpha$, and $\pi(p^*, q^*) - \hat{\pi}(\hat{p}) = \dfrac{(ad-bc)^2}{4bd(b+d)} \geq 0$. Note that the difference is 0 when $ad = bc$, in which case $p^* = q^*$, so the firm wants to charge the same price in each market anyway.

3. Imposing a tax of t per unit sold in market area 1 means that the new profit function is $\widehat{\pi}(Q_1, Q_2) = \pi(Q_1, Q_2) - tQ_1$. The optimal choice of production in market area 1 is then $\widehat{Q}_1 = (a_1 - \alpha - t)/2b_1$ (see the text), and the tax revenue is $T(t) = t(a_1 - \alpha - t)/2b_1 = [t(a_1 - \alpha) - t^2]/2b_1$. This quadratic polynomial has maximum when $T'(t) = 0$, so $t = \frac{1}{2}(a_1 - \alpha)$.

4. (a) Let $(x_0, y_0) = (0, 11.29)$, $(x_1, y_1) = (1, 11.40)$, $(x_2, y_2) = (2, 11.49)$, and $(x_3, y_3) = (3, 11.61)$, so that x_0 corresponds to 1970, etc. (The numbers y_t are approximate, as are most subsequent results.) We find that $\mu_x = \frac{1}{4}(0+1+2+3) = 1.5$, $\mu_y = \frac{1}{4}(11.29 + 11.40 + 11.49 + 11.61) = 11.45$, and $\sigma_{xx} = \frac{1}{4}[(0-1.5)^2 + (1-1.5)^2 + (2-1.5)^2 + (3-1.5)^2] = 1.25$. Moreover, we find $\sigma_{xy} = 0.13125$, and so $\hat{a} = \sigma_{xy}/\sigma_{xx} = 0.105$ and $\hat{b} = \mu_y - \hat{a}\mu_x \approx 11.45 - 0.105 \cdot 1.5 = 11.29$.

(b) With $z_0 = \ln 274$, $z_1 = \ln 307$, $z_2 = \ln 436$, and $z_3 = \ln 524$, we have $(x_0, z_0) = (0, 5.61)$, $(x_1, z_1) = (1, 5.73)$, $(x_2, z_2) = (2, 6.08)$, and $(x_3, z_3) = (3, 6.26)$. As before, $\mu_x = 1.5$ and $\sigma_{xx} = 1.25$. Moreover, $\mu_z = \frac{1}{4}(5.61 + 5.73 + 6.08 + 6.26) = 5.92$ and $\sigma_{xz} \approx 0.2875$. Hence $\hat{c} = \sigma_{xz}/\sigma_{xx} = 0.23$, $\hat{d} = \mu_z - \hat{c}\mu_x = 5.92 - 0.23 \cdot 1.5 = 5.575$.

(c) With ln (GNP) = $0.105x + 11.25$, GNP = $e^{11.25}e^{0.105x} = 80017e^{0.105x}$. Likewise, FA = $256e^{0.23x}$. The requirement that FA = 0.01 GNP implies that $e^{0.23x - 0.105x} = 80017/25600$, and so $0.125x = \ln(80017/25600)$. Thus $x = \ln(80017/25600)/0.125 = 9.12$. Since $x = 0$ corresponds to 1970, the goal would have been reached in 1979.

5. (a) See the text. (b) Firm A's profit is now $\pi_A(p) = px - 5 - x = p(29 - 5p + 4q) - 5 - 29 + 5p - 4q = 34p - 5p^2 + 4pq - 4q - 34$, with q fixed. This quadratic polynomial is maximized at $p = p_A(q) = \frac{1}{5}(2q + 17)$. Likewise, firm B's profit is now $\pi_B(q) = qy - 3 - 2y = 28q - 6q^2 + 4pq - 8p - 35$, with p fixed. This quadratic polynomial is maximized at $q = q_B(p) = \frac{1}{3}(p + 7)$. For (c) and (d) see the text.

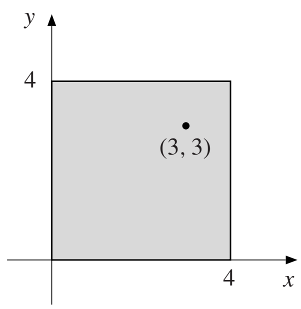


Figure SM13.5.2

13.5

2. (a) The continuous function f is defined on a closed, bounded set S (see Fig. SM13.5.2), so the extreme value theorem ensures that f attains both a maximum and a minimum over S. Stationary points are where

(i) $f_1'(x,y) = 3x^2 - 9y = 0$ and (ii) $f_2'(x,y) = 3y^2 - 9x = 0$. From (i), $y = \frac{1}{3}x^2$, which inserted into (ii) yields $\frac{1}{3}x(x^3 - 27) = 0$. The only solutions are $x = 0$ and $x = 3$. Thus the only stationary point in the interior of S is $(x, y) = (3, 3)$. We proceed by examining the behaviour of $f(x, y)$ along the boundary of S, i.e. along the four edges of S.
(I) $y = 0$, $x \in [0, 4]$. Then $f(x, 0) = x^3 + 27$, which has minimum at $x = 0$, and maximum at $x = 4$.
(II) $x = 4$, $y \in [0, 4]$. Then $f(4, y) = y^3 - 36y + 91$. The function $g(y) = y^3 - 36y + 91$, $y \in [0, 4]$ has $g'(y) = 3y^2 - 36 = 0$ at $y = \sqrt{12}$. Possible extreme points along (II) are therefore $(4, 0)$, $(4, \sqrt{12})$, and $(4, 4)$.
(III) $y = 4$, $x \in [0, 4]$. Then $f(x, 4) = x^3 - 36x + 91$, and as in (II) we see that possible extreme points are $(0, 4)$, $(\sqrt{12}, 4)$, and $(4, 4)$.
(IV) $x = 0$, $y \in [0, 4]$. As in case (I) we obtain the possible extreme points $(0, 0)$ and $(0, 4)$.

This results in six candidates, and $f(3, 3) = 0$, $f(0, 0) = 27$, $f(4, 0) = f(0, 4) = 91$, $f(4, \sqrt{12}) = f(\sqrt{12}, 4) = 91 - 24\sqrt{12} \approx 7.7$, $f(0, 0) = 27$. The conclusion follows.

(b) The constraint set $S = \{(x, y) : x^2 + y^2 \le 1\}$ are all points that lie on or inside a circle around the origin with radius 1. This is a closed and bounded set, and $f(x, y) = x^2 + 2y^2 - x$ is continuous. Therefore the extreme value theorem ensures that f attains both a maximum and a minimum over S.

Stationary points for f where $f_x'(x, y) = 2x - 1 = 0$ and $f_y'(x, y) = 4y = 0$. So the only stationary point for f is $(x_1, y_1) = (1/2, 0)$, which is an interior point of S.

An extreme point that does not lie in the interior of S must lie on the boundary of S, that is, on the circle $x^2 + y^2 = 1$. *Along this circle* we have $y^2 = 1 - x^2$, and therefore

$$f(x, y) = x^2 + 2y^2 - x = x^2 + 2(1 - x^2) - x = 2 - x - x^2$$

where x runs through the interval $[-1, 1]$. (It is a common error to ignore this restriction.) The function $g(x) = 2 - x - x^2$ has one stationary point in the interior of $[-1, 1]$, namely $x = -1/2$, so any extreme values of $g(x)$ must occur either for this value of x or at one the endpoints ± 1 of the interval $[-1, 1]$. Any extreme points for $f(x, y)$ on the boundary of S must therefore be among the points

$$(x_2, y_2) = (-\tfrac{1}{2}, \tfrac{1}{2}\sqrt{3}),\ (x_3, y_3) = (-\tfrac{1}{2}, -\tfrac{1}{2}\sqrt{3}),\ (x_4, y_4) = (1, 0),\ (x_5, y_5) = (-1, 0)$$

Now, $f(\frac{1}{2}, 0) = -\frac{1}{4}$, $f(-\frac{1}{2}, \pm\frac{1}{2}\sqrt{3}) = \frac{9}{4}$, $f(1, 0) = 0$, and $f(-1, 0) = 2$. The conclusion follows.

3. The set S is shown in Fig. A13.5.3 in the book. It is clearly closed and bounded, so the continuous function f has a maximum in S. The stationary points are where $\partial f/\partial x = 9 - 12(x + y) = 0$ and $\partial f/\partial y = 8 - 12(x + y) = 0$. But $12(x + y) = 9$ and $12(x + y) = 8$ give a contradiction. Hence, there are no stationary points at all. The maximum value of f must therefore occur on the boundary, which consists of five parts. Either the maximum value occurs at one of the five corners or "extreme points" of the boundary, or else at an interior point of one of five straight "edges." The function values at the five corners are $f(0, 0) = 0$, $f(5, 0) = -105$, $f(5, 3) = -315$, $f(4, 3) = -234$, and $f(0, 1) = 2$.

We proceed to examine the behaviour of f at interior points of each of the five edges in Fig. A13.5.3.
(I) $y = 0$, $x \in (0, 5)$. The behaviour of f is determined by the function $g_1(x) = f(x, 0) = 9x - 6x^2$ for $x \in (0, 5)$. If this function of one variable has a maximum in $(0, 5)$, it must occur at a stationary point where $g_1'(x) = 9 - 12x = 0$, and so at $x = 3/4$. We find that $g_1(3/4) = f(3/4, 0) = 27/8$.
(II), $x = 5$, $y \in (0, 3)$. Define $g_2(y) = f(5, y) = 45 + 8y - 6(5 + y)^2$ for $y \in (0, 3)$. Here $g_2'(y) = -52 - 12y$, which is negative throughout $(0, 3)$, so there are no stationary points on this edge.

(III) $y = 3$, $x \in (4, 5)$. Define $g_3(x) = f(x, 3) = 9x + 24 - 6(x + 3)^2$ for $x \in (4, 5)$. Here $g_3'(x) = -27 - 12x$, which is negative throughout $(4, 5)$, so there are no stationary points on this edge either.
(IV) $-x + 2y = 2$, or $y = x/2 + 1$, with $x \in (0, 4)$. Define the function $g_4(x) = f(x, x/2 + 1) = -27x^2/2 - 5x + 2$ for $x \in (0, 4)$. Here $g_4'(x) = -27x - 5$, which is negative in $(0, 4)$, so there are no stationary points here.
(V) $x = 0$, $y \in (0, 1)$. Define $g_5(y) = f(0, y) = 8y - 6y^2$. Then $g_5'(y) = 8 - 12y = 0$ at $y = 2/3$, with $g_5(2/3) = f(0, 2/3) = 8/3$.

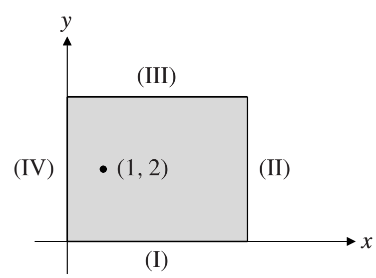
After comparing the values of f at the five corners of the boundary and at the points found on the edges labeled (I) and (V), we conclude that the maximum value of f is 27/8, which is achieved at $(3/4, 0)$.

5. (a) $f_1'(x, y) = e^{-x}(1 - x)(y - 4)y$, $f_2'(x, y) = 2xe^{-x}(y - 2)$. It follows that the stationary points are $(1, 2)$, $(0, 0)$ and $(0, 4)$. (Make sure you understand why!) Moreover, $f_{11}''(x, y) = e^{-x}(x - 2)(y^2 - 4y)$, $f_{12}''(x, y) = e^{-x}(1 - x)(2y - 4)$, and $f_{22}'' = 2xe^{-x}$. Classification of the stationary points:

(x, y)	A	B	C	$AC - B^2$	Type of point
$(1, 2)$	$4e^{-1} > 0$	0	$2e^{-1}$	$8e^{-2} > 0$	Loc. min. point
$(0, 0)$	0	-4	0	$-16 < 0$	Saddle point
$(0, 4)$	0	4	0	$-16 < 0$	Saddle point

(b) We see that $f(1, y) = e^{-1}(y^2 - 4y) \to \infty$ as $y \to \infty$. This shows that f has no global maximum point. Since $f(-1, y) = -e(y^2 - 4y) \to -\infty$ as $y \to \infty$, f has no global minimum point either.

(c) The set S is obviously bounded. The boundary of S consists of the four edges of the rectangle, and all points on these line segments belong to S. Hence S is closed. Since f is continuous, the extreme value theorem tells us that f has global maximum and minimum points in S. These global extreme points must be either stationary points if f in the interior of S, or points on the boundary of S. The only stationary point of f in the interior of S is $(1, 2)$. The function value at this point is $f(1, 2) = -4e^{-1} \approx 1.4715$.



The four edges are most easily investigated separately:
(i) Along (I), $y = 0$ and $f(x, y) = f(x, 0)$ is identically 0.
(ii) Along (II), $x = 5$ and $f(x, y) = 5e^{-5}(y^2 - 4y)$, which has its least value for $y = 2$ and its greatest value for $y = 0$ and for $y = 4$. (Note that $y \in [0, 4]$ for all points (x, y) on the line segment (II).) The values are $f(5, 2) = -20e^{-5} \approx -0.1348$ and $f(5, 0) = f(5, 4) = 0$.
(iii) On edge (III), $y = 4$ and $f(x, y) = f(x, 4) = 0$.
(iv) Finally, along (IV), $x = 0$ and $f(x, y) = f(0, y) = 0$.

Collecting all these results, we see that f attains its least value (on S) at the point $(1, 2)$ and its greatest value (namely 0) at all points of the line segments (I), (III) and (IV).

(d) $y' = -\dfrac{f_1'(x, y)}{f_2'(x, y)} = -\dfrac{e^{-x}(1 - x)(y - 4)y}{2xe^{-x}(y - 2)} = \dfrac{(x - 1)(y - 4)y}{2x(y - 2)} = 0$ when $x = 1$.

13.6

1. (a) $f_x'(x, y, z) = 2 - 2x = 0$, $f_y'(x, y, z) = 10 - 2y = 0$, $f_x'(x, y, z) = -2z = 0$. The conclusion follows.

(b) $f'_x(x, y, z) = -2x - 2y - 2z = 0$, $f'_y(x, y, z) = -4y - 2x = 0$, $f'_z(x, y, z) = -6z - 2x = 0$. From the two last equations we get $y = -\frac{1}{2}x$ and $z = -\frac{1}{3}x$. Inserting this into the first equation we get $-2x + x + \frac{2}{3}x = 0$, and thus $x = 0$, and then $y = z = 0$.

4. To calculate f'_x is routine. To differentiate f w.r.t. y and z we use (9.3.7) and (9.3.6). The derivative of $\int_y^z e^{t^2}\,dt$ w.r.t. y, keeping z constant is according to (9.3.7), $-e^{y^2}$. The derivative of $\int_y^z e^{t^2}\,dt$ w.r.t. z, keeping y constant is according to (9.3.6), e^{z^2}. Thus $f'_y = 2 - e^{y^2}$ and $f'_z = -3 + e^{z^2}$. Since each of the three partials depends only on one variable and is 0 for two different values of that variable, there are eight stationary point. See the text.

13.7

2. (a) First-order conditions: $\pi'_K = \frac{2}{3}pK^{-1/3} - r = 0$, $\pi'_L = \frac{1}{2}pL^{-1/2} - w = 0$, $\pi'_T = \frac{1}{3}pT^{-2/3} - q = 0$. Thus, $K^{-1/3} = 3r/2p$, $L^{-1/2} = 2w/p$, and $T^{-2/3} = 3q/p$. Raising each side of $K^{-1/3} = 3r/2p$ to the power of -3 yields, $K = (3r/2p)^{-3} = (2p/3r)^3 = (8/27)p^3r^{-3}$. In a similar way we find L, and T. (b) See the text.

4. Differentiating $pF'_K(K^*, L^*) = r$ using the product rule gives $dp\,F'_K(K^*, L^*) + pd(F'_K(K^*, L^*)) = dr$. Moreover, $d(F'_K(K^*, L^*)) = F''_{KK}(K^*, L^*)\,dK^* + F''_{KL}(K^*, L^*)\,dL^*$. (To see why, note that $dg(K^*, L^*) = g'_K(K^*, L^*)\,dK^* + g'_L(K^*, L^*)\,dL^*$. Then let $g = F'_K$.) This explains the first displayed equation (replacing dK by dK^* and dL by dL^*). The second is derived in the same way.
(b) Rearrange the equation system by moving the differentials of the exogenous variables to the right-hand side, suppressing the fact that the partials are evaluated at (K^*, L^*):

$$pF''_{KK}\,dK^* + pF''_{KL}\,dL^* = dr - F'_K\,dp$$
$$pF''_{LK}\,dK^* + pF''_{LL}\,dL^* = dw - F'_L\,dp$$

We use Cramer's rule to express the differentials dK^* and dL^* in terms of dp, dr, and dw. Putting $\Delta = F''_{KK}F''_{LL} - F''_{KL}F''_{LK} = F''_{KK}F''_{LL} - (F''_{KL})^2$, we get

$$dK^* = \frac{1}{p^2\Delta}\begin{vmatrix} dr - F'_K\,dp & pF''_{KL} \\ dw - F'_L\,dp & pF''_{LL} \end{vmatrix} = \frac{-F'_KF''_{LL} + F'_LF''_{KL}}{p\Delta}\,dp + \frac{F''_{LL}}{p\Delta}\,dr + \frac{-F''_{KL}}{p\Delta}\,dw$$

In the same way

$$dL^* = \frac{1}{p^2\Delta}\begin{vmatrix} pF''_{KK} & dr - F'_K\,dp \\ pF''_{LK} & dw - F'_L\,dp \end{vmatrix} = \frac{-F'_LF''_{KK} + F'_KF''_{LK}}{p\Delta}\,dp + \frac{-F''_{LK}}{p\Delta}\,dr + \frac{F''_{KK}}{p\Delta}\,dw$$

We can now read off the required partials. (Note that there is an error in the expression for $\partial K^*/\partial p$ in the answer to this problem in the text. In the numerator, replace F''_{KK} by F''_{LL}. (c) See the text. (Recall that $F''_{LL} < 0$ follows from (∗∗) in Example 13.3.3.)

5. (a) (i) $R'_1(x_1^*, x_2^*) + s = C'_1(x_1^*, x_2^*)$ (marginal revenue plus subsidy equal marginal cost)
(ii) $R'_2(x_1^*, x_2^*) = C'_2(x_1^*, x_2^*) + t = 0$ (marginal revenue equals marginal cost plus tax). For (b) see the text. (c) Taking the total differentials of (i) and (ii) yields

$$(R''_{11} - C''_{11})dx_1^* + (R''_{12} - C''_{12})dx_2^* = -ds, \quad (R''_{21} - C''_{21})dx_1^* + (R''_{22} - C''_{22})dx_2^* = dt$$

Solving for dx_1^* and dx_2^* yields, after rearranging,

$$dx_1^* = \frac{-(R''_{22} - C''_{22})ds - (R''_{12} - C''_{12})dt}{D}, \quad dx_2^* = \frac{(R''_{21} - C''_{21})ds + (R''_{11} - C''_{11})dt}{D}$$

From this we find that the partial derivatives are

$$\frac{\partial x_1^*}{\partial s} = \frac{-R''_{22} + C''_{22}}{D} > 0, \ \frac{\partial x_1^*}{\partial t} = \frac{-R''_{12} + C''_{12}}{D} > 0, \ \frac{\partial x_2^*}{\partial s} = \frac{R''_{21} - C''_{21}}{D} < 0, \ \frac{\partial x_2^*}{\partial t} = \frac{R''_{11} - C''_{11}}{D} < 0$$

where the signs follow from the assumptions in the problem and the fact that $D > 0$ from (b). Note that these signs accord with economic intuition. For example, if the tax on good 2 increases, then the production of good 1 increases, while the production of good 2 decreases.
(d) Follows from the expressions in (c) because $R''_{12} = R''_{21}$ and $C''_{12} = C''_{21}$.

Review Problems for Chapter 13

2. (a) The profit function is $\pi(Q_1, Q_2) = 120Q_1 + 90Q_2 - 0.1Q_1^2 - 0.1Q_1Q_2 - 0.1Q_2^2$. First-order conditions for maximal profit are: $\pi'_1(Q_1, Q_2) = 120 - 0.2Q_1 - 0.1Q_2 = 0$ and $\pi'_2(Q_1, Q_2) = 90 - 0.1Q_1 - 0.2Q_2 = 0$. We find $(Q_1, Q_2) = (500, 200)$. Moreover, $\pi''_{11}(Q_1, Q_2) = -0.2 \le 0$, $\pi''_{12}(Q_1, Q_2) = -0.1$, and $\pi''_{22}(Q_1, Q_2) = -0.2 \le 0$. Since also $\pi''_{11}\pi''_{22} - (\pi''_{12})^2 = 0.03 \ge 0$, $(500, 200)$ maximizes profits.
(b) The profit function is now, $\hat{\pi}(Q_1, Q_2) = P_1Q_1 + 90Q_2 - 0.1Q_1^2 - 0.1Q_1Q_2 - 0.1Q_2^2$. First-order conditions for maximal profit: $\hat{\pi}_1 = P_1 - 0.2Q_1 - 0.1Q_2 = 0, \hat{\pi}_2 = 90 - 0.1Q_1 - 0.2Q_2 = 0$. If we need to have $Q_1 = 400$, the first-order conditions reduce to $P_1 - 80 - 0.1Q_2 = 0$ and $90 - 40 - 0.2Q_2 = 0$. It follows that $P_1 = 105$.

3. (a) Stationary points where $P'_1(x, y) = -0.2x - 0.2y + 47 = 0$ and $P'_2(x, y) = -0.2x - 0.4y + 48y = 0$. It follows that $x = 230$ and $y = 5$. Moreover, $P''_{11} = -0.2 \le 0$, $P''_{12} = -0.2$, and $P''_{22} = -0.4 \le 0$. Since also $P''_{11}P''_{22} - (P''_{12})^2 = 0.04 \ge 0$, $(230, 5)$ maximizes profits. (b) With $x + y = 200$, and so $y = 200 - x$, the new profit function is $\hat{\pi}(x) = f(x, 200 - x) = -0.1x^2 + 39x + 1000$. This function is easily seen to have maximum at $x = 195$. Then $y = 200 - 195 = 5$.

4. (a) Stationary points:(i) $f'_1(x, y) = 3x^2 - 2xy = x(3x - 2y) = 0$, (ii) $f'_2(x, y) = -x^2 + 2y = 0$. From (i), $x = 0$ or $3x = 2y$. If $x = 0$, then (ii) gives $y = 0$. If $3x = 2y$, then (ii) gives $3x = x^2$, and so $x = 0$ or $x = 3$. If $x = 3$, then (ii) gives $y = x^2/2 = 9/2$. So the stationary points are $(0, 0)$ and $(3, 9/2)$.
(b) (i) $f'_1(x, y) = ye^{4x^2-5xy+y^2}(8x^2 - 5xy + 1) = 0$, (ii) $f'_2(x, y) = xe^{4x^2-5xy+y^2}(2y^2 - 5xy + 1) = 0$. If $y = 0$, then (i) is satisfied and (ii) is only satisfied when $x = 0$. If $x = 0$, then (ii) is satisfied and (i) is only satisfied when $y = 0$. Thus, in addition to $(0, 0)$, any other stationary point must satisfy $8x^2 - 5xy + 1 = 0$ and $2y^2 - 5xy + 1 = 0$. Subtracting the second from the first yields $8x^2 = 2y^2$, or $y = \pm 2x$. Inserting $y = -2x$ into $8x^2 - 5xy + 1 = 0$ yields $18x^2 + 1 = 0$, which has no solutions. Inserting $y = 2x$ into $8x^2 - 5xy + 1 = 0$ yields $x^2 = \frac{1}{2}$, and so $x = \pm\frac{1}{2}\sqrt{2}$. We conclude that the stationary points are: $(0, 0)$ and $(\frac{1}{2}\sqrt{2}, \sqrt{2})$, $(-\frac{1}{2}\sqrt{2}, -\sqrt{2})$

5. (a) The first-order conditions for (K^*, L^*, T^*) to maximize π are: $\pi'_K = pa/K^* - r = 0$, $\pi'_L = pb/L^* - w = 0$, $\pi'_T = pc/T^* - q = 0$. Hence, $K^* = ap/r$, $L^* = bp/w$, $T^* = cp/q$.
(b) $\pi^* = pa\ln(ap) - pa\ln r + pb\ln(bp/w) + pc\ln(cp/q) - ap - bp - cp = -pa\ln r$ plus terms that do not depend on r. So $\partial\pi^*/\partial r = -pa/r = -K^*$. (c) See the text.

7. (a) $f_1'(x, y) = 2x - y - 3x^2$, $f_2'(x, y) = -2y - x$, $f_{11}''(x, y) = 2 - 6x$, $f_{12}''(x, y) = -1$, $f_{22}''(x, y) = -2$. Stationary points where $2x - y - 3x^2 = 0$ and $-2y - x = 0$. The last equation yields $y = -x/2$, which inserted into the second equation gives $x(\frac{5}{6} - x) = 0$. It follows that there are two stationary points, $(x_1, y_1) = (0, 0)$ and $(x_2, y_2) = (5/6, -5/12)$. These points are classified in the following table:

(x, y)	A	B	C	$AC - B^2$	Type of point
$(0, 0)$	2	-1	-2	-5	Saddle point
$(\frac{5}{6}, -\frac{5}{12})$	-3	-1	-2	5	Lok. max. point

(b) f is concave in the domain where $f_{11}'' \leq 0$, $f_{22}'' \leq 0$, and $f_{11}''f_{22}'' - (f_{12}'')^2 \geq 0$, i.e. where $2 - 6x \leq 0$, $-2 \leq 0$, and $(2 - 6x)(-2) - (-1)^2 \geq 0$. These conditions are equivalent to $x \geq 1/3$ and $x \geq 5/12$. In particular, one must $x \geq 5/12$. Since $5/12 > 1/3$, f is concave in the set S consisting of all (x, y) where $x \geq 5/12$.
(c) The stationary point $(x_2, y_2) = (5/6, -5/12)$ found in (a) does belong to S. Since f concave in S, this is a (global) maximum point for f in S. $f_{\max} = 125/432$.

8. (a) We find $f_1'(x, y) = x - 1 + ay$, $f_2'(x, y) = a(x - 1) - y^2 + 2a^2y$. Stationary points require that $x - 1 = -ay$ and $a(x - 1) = y^2 - 2a^2y$. These two equations yield $-a^2y = y^2 - 2a^2y$, and so $a^2y = y^2$. Hence $y = 0$ or $y = a^2$. Since $x = 1 - ay$, the stationary points are $(1, 0)$ and $(1 - a^3, a^2)$. (Since we were asked only to show that $(1 - a^3, a^2)$ is a stationary point, it would suffice to verify that it makes both partials equal to 0.) (b) Note that the partial derivative of f w.r.t. a, keeping x and y constant is $\partial f/\partial a = y(x - 1) + 2ay^2$. Evaluated at $x = 1 - a^3$, $y = a^2$, this partial derivative is also a^5, thus confirming the envelope theorem. (c) See the text.

9. For (a)–(c), see the text. (d) $\partial^2\hat{\pi}/\partial p^2 = -2b$, $\partial^2\hat{\pi}/\partial q^2 = -2\gamma$, and $\partial^2\hat{\pi}/\partial p\partial y = \beta + c$. The direct partials of order 2 are negative and $\Delta = (\partial^2\hat{\pi}/\partial p^2)(\partial^2\hat{\pi}/\partial q^2) - (\partial^2\hat{\pi}/\partial p\partial q)^2 = 4\gamma b - (\beta + c)^2$, so the conclusion follows.

14 Constrained Optimization

14.1

4. (a) With $\mathcal{L}(x, y) = x^2 + y^2 - \lambda(x + 2y - 4)$, the first-order conditions are $\mathcal{L}_1' = 2x - \lambda = 0$ and $\mathcal{L}_2' = 2y - 2\lambda = 0$. From these equations we get $2x = y$, which inserted into the constraint gives $x + 4x = 4$. So $x = 4/5$ and $y = 2x = 8/5$, with $\lambda = 2x = 8/5$.
(b) The same method as in (a) gives $2x - \lambda = 0$ and $4y - \lambda = 0$, so $x = 2y$. From the constraint we get $x = 8$ and $y = 4$, with $\lambda = 16$. (c) The first-order conditions imply that $2x + 3y = \lambda = 3x + 2y$, which implies $x = y$. So the solution is $(x, y) = (50, 50)$ with $\lambda = 250$.

5. The budget constraint is $2x + 4y = 1000$, so with $\mathcal{L}(x, y) = 100xy + x + 2y - \lambda(2x + 4y - 1000)$, the first-order conditions are $\mathcal{L}_1' = 100y + 1 - 2\lambda = 0$ and $\mathcal{L}_2' = 100x + 2 - 4\lambda = 0$. From these equations, by eliminating λ, we get $x = 2y$, which inserted into the constraint gives $2x + 2x = 1000$. So $x = 250$ and $y = 125$.

7. The problem is: max $-0.1x^2 - 0.2xy - 0.2y^2 + 47x + 48y - 600$ subject to $x + y = 200$. With $\mathcal{L}(x, y) = -0.1x^2 - 0.2xy - 0.2y^2 + 47x + 48y - 600 - \lambda(x + y - 200)$, the first-order conditions are $\mathcal{L}'_1 = -0.2x - 0.2y + 47 - \lambda = 0$ and $\mathcal{L}'_2 = -0.2x - 0.4y + 48 - \lambda = 0$. Eliminating λ yields $y = 5$, and then the budget constraint gives $x = 195$.

9. (a) With $\mathcal{L}(x, y) = 100 - e^{-x} - e^{-y} - \lambda(px + qy - m)$, $\mathcal{L}'_x = \mathcal{L}'_y = 0$ when $e^{-x} = \lambda p$ and $e^{-y} = \lambda q$. Hence, $x = -\ln(\lambda p) = -\ln\lambda - \ln p$, $y = -\ln\lambda - \ln q$. Inserting these expressions for x and y into the constraint, then solving for $\ln\lambda$, yields $\ln\lambda = -(m + p\ln p + q\ln q)/(p+q)$. Therefore $x(p, q, m) = [m + q\ln(q/p)]/(p+q)$, $y(p, q, m) = [m + p\ln(p/q)]/(p+q)$.
(b) $x(tp, tq, tm) = [tm + tq\ln(tq/tp)]/(tp + tq) = x(p, q, m)$, so x is homogeneous of degree 0. In the same way we see that $y(p, q, m)$ is homogeneous of degree 0.

14.2

3. (a) Solving $x + 2y = a$ for y yields $y = \frac{1}{2}a - \frac{1}{2}x$, and then $x^2 + y^2 = x^2 + (\frac{1}{2}a - \frac{1}{2}x)^2 = \frac{5}{4}x^2 - \frac{1}{2}ax + \frac{1}{4}a^2$. This quadratic function certainly has a minimum at $x = a/5$. (b) $\mathcal{L}(x, y) = x^2 + y^2 - \lambda(x + 2y - a)$. The necessary conditions are $\mathcal{L}'_1 = 2x - \lambda = 0$, $\mathcal{L}'_2 = 2y - 2\lambda = 0$, implying that $2x = y$. From the constraint, $x = a/5$ and then $y = 2a/5$, $\lambda = 2a/5$.

The value function $f^*(a) = (a/5)^2 + (2a/5)^2 = a^2/5$, so $df^*(a)/da = 2a/5$, which is also the value of the Lagrangian multiplier. Equation (2) is confirmed. (c) See the text.

4. (a) With $\mathcal{L}(x, y) = \sqrt{x} + y - \lambda(x + 4y - 100)$, the first-order conditions for (x^*, y^*) to solve the problem are: (i) $\partial\mathcal{L}/\partial x = 1/2\sqrt{x^*} - \lambda = 0$ (ii) $\partial\mathcal{L}/\partial y = 1 - 4\lambda = 0$. From (ii), $\lambda = 1/4$, which inserted into (i) yields $\sqrt{x^*} = 2$, so $x^* = 4$. Then $y^* = 25 - \frac{1}{4}4 = 24$, and maximal utility is $U^* = \sqrt{x^*} + y^* = 26$.
(b) Denote the new optimal values of x and y by $\hat{x}$ and $\hat{y}$. If 100 is changed to 101, still $\lambda = 1/4$ and $\hat{x} = 4$. The constraint now gives $4 + 4\hat{y} = 101$, so that $\hat{y} = 97/4 = 24.25$, with $\hat{U} = \sqrt{\hat{x}} + \hat{y} = 26.25$. The increase in the maximum utility as 100 is increased to 101, is thus $\hat{U} - U^* = 0.25 = \lambda$. (In general, the increase in utility is *approximately* equal to the value of the Lagrange multiplier.)
(c) The necessary conditions for optimality are now $\partial\mathcal{L}/\partial x = 1/2\sqrt{x^*} - \lambda p = 0$, $\partial\mathcal{L}/\partial y = 1 - \lambda q = 0$. Proceeding in the same way as in (a), we find $\lambda = 1/q$, $\sqrt{x^*} = q/2p$, and so $x^* = q^2/4p^2$, with $y^* = m/q - q/4p$. (Note that $y^* > 0 \iff m > q^2/4p$.) (If we solve the constraint for y, the utility function is $u(x) = \sqrt{x} + (m - px)/q$. We see that $u'(x) = 1/2\sqrt{x} - p/q = 0$ for $x^* = q^2/4p^2$ and $u''(x) = -(1/4)x^{-3/2} < 0$ when $x > 0$. So we have found the maximum.)

5. (a) $px^* = pa + \alpha/\lambda$ and $qy^* = qb + \beta/\lambda$ give $m = px^* + qy^* = pa + qb + (\alpha + \beta)/\lambda = pa + qb + 1/\lambda$, so $1/\lambda = m - (pa + qb)$. The expressions given in $(**)$ are now easily established. (If we think of a and b as kind of existence minimum of the two goods, the assumption $pa + qb < m$ means that the consumer can afford to buy (a, b).) (b) With the U^* given in the answer, since $\alpha + \beta = 1$,
$\dfrac{\partial U^*}{\partial m} = \dfrac{\alpha}{m - (pa+qb)} + \dfrac{\beta}{m - (pa+qb)} = \dfrac{1}{m - (pa+qb)} = \lambda > 0$. Moreover,
$\dfrac{\partial U^*}{\partial p} = \dfrac{-\alpha a}{m - (pa+qb)} - \dfrac{\alpha}{p} + \dfrac{-\beta a}{m - (pa+qb)} = \dfrac{-a}{m - (pa+qb)} - \dfrac{\alpha}{p} = -a\lambda - \dfrac{\alpha}{p}$, and $-\dfrac{\partial U^*}{\partial m}x^* = -\lambda(a + \dfrac{\alpha}{\lambda p}) = -a\lambda - \dfrac{\alpha}{p}$, so $\dfrac{\partial U^*}{\partial p} = -\dfrac{\partial U^*}{\partial m}x^*$. The last equality is shown in the same way.

6. $f(x, T) = x\int_0^T [-t^3 + (\alpha T^2 + T - 1)t^2 + (T - \alpha T^3)t]\,dt = x\Big|_0^T [-\frac{1}{4}t^4 + (\alpha T^2 + T - 1)\frac{1}{3}t^3 + (T - \alpha T^3)\frac{1}{2}t^2] = -\frac{1}{6}\alpha x T^5 + \frac{1}{12}xT^4 + \frac{1}{6}xT^3$. In the same way, $g(x, T) = \frac{1}{6}xT^3$. The solution of $(*)$ is $x = 384\alpha^3 M$, $T = 1/4\alpha$, with $f^*(M) = M + M/16\alpha$. (The easiest way to solve the problem is to note that because $\frac{1}{6}xT^3 = M$, the problem reduces to that of maximizing $M + \frac{1}{2}MT - \alpha MT^2$ for $T \geq 0$. For $T = 0$, the expression is equal to M, and its maximum is attained for $T = 1/4\alpha$.)

Alternatively, eliminating the Lagrange multiplier from the first-order conditions $f_1' = \lambda g_1'$ and $f_2' = \lambda g_2'$, we eventually obtain $T = 1/4\alpha$. The Lagrange multiplier is $\lambda = 1 + 1/16\alpha$. Clearly, $\partial f^*(M)/\partial M = \lambda$, which confirms (2).

14.3

1. (a) With $\mathcal{L}(x, y) = 3xy - \lambda(x^2 + y^2 - 8)$, the first-order conditions are $\mathcal{L}_1' = 3y - 2\lambda x = 0$ and $\mathcal{L}_2' = 3x - 2\lambda y = 0$. Since (0, 0) does not satisfy the constraint, from these equations we get $x^2 = y^2$. Inserted into the constraint this yields $x^2 = 4$, and so $x = \pm 2$, and the solution candidates are: $(2, 2)$, $(2, -2)$, $(-2, 2)$, $(-2, -2)$. Here $f(2, 2) = f(-2, -2) = 12$ and $f(-2, 2) = f(2, -2) = -12$. So $(2, 2)$ and $(-2, -2)$ solves the maximization problem, and $(-2, 2)$ and $(2, -2)$ solves the minimization problem, because the extreme value theorem ensures that solutions exist. (f is continuous and the constraint curve is a closed bounded set (a circle).)
(b) With $\mathcal{L} = x + y - \lambda(x^2 + 3xy + 3y^2 - 3)$, the first-order conditions are $1 - 2\lambda x - 3\lambda y = 0$ and $1 - 3\lambda x - 6\lambda y = 0$. From these equations we get $2\lambda x + 3\lambda y = 3\lambda x + 6\lambda y$, or $\lambda(3y + x) = 0$. Here $\lambda = 0$ is impossible, so $x = -3y$. Inserted into the constraint we have $(3, -1)$ and $(-3, 1)$ as the only possible solutions of the maximization and minimization problems, respectively. The extreme value theorem ensures that solutions exist. (The objective function is continuous and the constraint curve is a closed bounded set (an ellipse, see (5.5.5)).

2. (a) With $\mathcal{L} = x^2+y^2-2x+1-\lambda(x^2+4y^2-16)$, the first-order conditions are (i) $2x-2-2\lambda x = 0$ and (ii) $2y - 8\lambda y = 0$. Equation (i) yields $\lambda = 1 - 1/x$ (why can we be sure that $x \neq 0$?), and equation (ii) shows that $y = 0$ or $\lambda = 1/4$. If $y = 0$, then $x^2 = 16-4y^2 = 16$, so $x = \pm 4$, and then gives $\lambda = 1 \mp 1/4$. If $y \neq 0$, then $\lambda = 1/4$ and (i) gives $2x - 2 - x/2 = 0$, so $x = 4/3$. The constraint $x^2 + 4y^2 = 16$ now yields $4y^2 = 16 - 16/9 = 128/9$, so $y = \pm\sqrt{32/9} = \pm 4\sqrt{2}/3$. Thus, there are four solution candidates: (i) $(x, y, \lambda) = (4, 0, 3/4)$, (ii) $(x, y, \lambda) = (-4, 0, 5/4)$, (iii) $(x, y, \lambda) = (4/3, 4\sqrt{2}/3, 1/4)$, and (iv) $(x, y, \lambda) = (4/3, -4\sqrt{2}/3, 1/4)$. Of these, the second is the maximum point (while (iii) and (iv) are the minimum points).
(b) The Lagrangian is $\mathcal{L} = \ln(2 + x^2) + y^2 - \lambda(x^2+2y-2)$. Hence, the necessary first-order conditions for (x, y) to be a minimum point are (i) $\partial\mathcal{L}/\partial x = 2x/(2 + x^2) - 2\lambda x = 0$ (ii) $\partial\mathcal{L}/\partial y = 2y - 2\lambda = 0$, (iii) $x^2 + 2y = 2$. From (i) we get $x\big(1/(2 + x^2) - \lambda\big) = 0$, so $x = 0$ or $\lambda = 1/(2 + x^2)$.
(I) If $x = 0$, then (iii) gives $y = 1$, so $(x_1, y_1) = (0, 1)$ is a candidate.
(II) If $x \neq 0$, then $y = \lambda = 1/(2 + x^2)$, where we used (ii). Inserting $y = 1/(2 + x^2)$ into (iii) gives $x^2 + 2/(2 + x^2) = 2 \iff 2x^2 + x^4 + 2 = 4 + 2x^2 \iff x^4 = 2 \iff x = \pm\sqrt[4]{2}$.
From (iii), $y = 1 - \frac{1}{2}x^2 = 1 - \frac{1}{2}\sqrt{2}$. Thus, $(x_2, y_2) = (\sqrt[4]{2}, 1 - \frac{1}{2}\sqrt{2})$ and $(x_3, y_3) = (-\sqrt[4]{2}, 1 - \frac{1}{2}\sqrt{2})$ are candidates. Now $f(x_1, y_1) = f(0, 1) = \ln 2 + 1 \approx 1.69$, $f(x_2, y_2) = f(x_3, y_3) = \ln(2 + \sqrt{2}) + (1 - \frac{1}{2}\sqrt{2})^2 = \ln(2 + \sqrt{2}) + \frac{3}{2} - \sqrt{2} \approx 1.31$. Hence, the minimum points for $f(x, y)$ (subject to $x^2 + 2y = 2$) are (x_2, y_2) and (x_3, y_3).

4. (a) With $\mathcal{L} = 24x - x^2 + 16y - 2y^2 - \lambda(x^2 + 2y^2 - 44)$, the first-order conditions are (i) $\mathcal{L}_1' = 24-2x-2\lambda x = 0$ and (ii) $\mathcal{L}_2' = 16-4y-4\lambda y = 0$. From (i) $x(1+\lambda) = 12$ and from (ii) $y(1+\lambda) = 4$. Eliminating λ from (i) and (ii) we get $x = 3y$. Inserted into the constraint, $11y^2 = 44$, so $y = \pm 2$, and then $x = \pm 6$. So there are two candidates, $(x, y) = (6, 2)$ and $(-6, -2)$, with $\lambda = 1$. Computing the objective function at these two points, the only possible solution is $(x, y) = (6, 2)$. Since the objective function is continuous and the constraint curve is closed and bounded (an ellipse), the extreme value theorem assures us that the optimum is found. (b) According to (14.2.3) the approximate change is $\lambda \cdot 1 = 1$.

14.4

4. The minimum is 1 at $(x, y) = (-1, 0)$. Actually, this problem is quite tricky. The Lagrangian can be written in the form $\mathcal{L} = (x+2)^2 + (1-\lambda)y^2 + \lambda x(x+1)^2$. The only stationary point which satisfies the constraint is $(0, 0)$, with $\lambda = -4$, and with $f(0, 0) = 4$. (In fact, $\mathcal{L}'_2 = 0$ only if $\lambda = 1$ or $y = 0$. For $\lambda = 1$, $\mathcal{L}'_1 = 3(x+1)^2 + 2 > 0$ for all x. For $y = 0$, the constraint gives $x = 0$ or $x = -1$. But $x = -1$ gives $\mathcal{L}'_1 = 2$, so $x = 0$ is necessary for a stationary point.) Yet at $(-1, 0)$ both $g'_1(-1, 0)$ and $g'_2(-1, 0)$ are 0, and the Lagrange multiplier method fails. The given problem is to minimize (the square of) the distance from $(-2, 0)$ to a point on the graph of $g(x, y) = 0$. But the graph consists of the isolated point $(-1, 0)$ and a smooth curve, as illustrated in Fig. SM14.4.4.

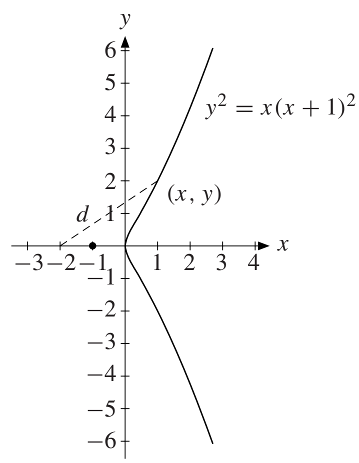


Figure SM14.4.4

14.5

4. $U''_{11}(x, y) = a(a-1)x^{a-2} \le 0$, $U''_{22}(x, y) = a(a-1)y^{a-2} \le 0$, and $U''_{12}(x, y) = 0$, so U is concave. With $\mathcal{L} = x^a + y^a - \lambda(px + qy - m)$, the first-order conditions are $\mathcal{L}'_1 = ax^{a-1} - \lambda p = 0$ and $\mathcal{L}'_2 = ay^{a-1} - \lambda q = 0$. Hence, $ax^{a-1} = \lambda p$ and $ay^{a-1} - \lambda q$. Eliminating λ we get $(x/y)^{a-1} = p/q$, and so $x = y(p/q)^{1/(a-1)}$. Inserted into the budget constraint we get
$px + qy = py(p/q)^{1/(a-1)} + qy = yp^{a/(a-1)}q^{-1/(a-1)} + qy = yq^{-1/(a-1)}[p^{a/(a-1)} + q^{a/(a-1)}] = m$,
and so $y = mq^{1/(a-1)}/[p^{a/(a-1)} + q^{a/(a-1)}]$. A similar expression is obtained for x.

14.6

1. (a) $\mathcal{L}'_x = 2x - \lambda = 0$, $\mathcal{L}'_y = 2y - \lambda = 0$, $\mathcal{L}'_z = 2z - \lambda = 0$. It follows that $x = y = z$, etc. See the text.

3. (a) The Lagrangian is $\mathcal{L} = \alpha \ln x + \beta \ln y + (1-\alpha-\beta)\ln(L-\ell) - \lambda(px + qy - w\ell - m)$, which is stationary when: (i) $\mathcal{L}'_x = \alpha/x^* - \lambda p = 0$; (ii) $\mathcal{L}'_y = \beta/y^* - \lambda q = 0$; (iii) $\mathcal{L}'_\ell = -(1-\alpha-\beta)/(L-\ell^*) + \lambda w = 0$. From (i) and (ii), $qy^* = (\beta/\alpha)px^*$, while (i) and (iii) yield $\ell^* w = wL - [(1-\alpha-\beta)/\alpha]px^*$. Insertion into the budget constraint and solving for x^* yields the answer in the text. The corresponding values for y^* and ℓ^* follows. The assumption $m \le [(1-\alpha-\beta)/\alpha]wL$ ensures that $\ell^* \ge 0$. (b) See the text.

6. The Lagrangian is $\mathcal{L} = x + y - \lambda(x^2 + 2y^2 + z^2 - 1) - \mu(x + y + z - 1)$, which is stationary when: (i) $\mathcal{L}'_x = 1 - 2\lambda x - \mu = 0$; (ii) $\mathcal{L}'_y = 1 - 4\lambda y - \mu = 0$; (iii) $\mathcal{L}'_z = -2\lambda z - \mu = 0$. From (i) and (ii), $\lambda(x-2y)=0$. If $\lambda=0$, then (ii) and (iii) yield the contradiction $\mu=1$ and $\mu=0$. Therefore $x=2y$ instead. Substituting this value for x into the constraints gives $6y^2+z^2=1$, $3y+z=1$. Thus $z=1-3y$ and $1=6y^2+(1-3y)^2=15y^2-6y+1$. Hence $y=0$ or $y=2/5$, implying that $x=0$ or 4/5, and that $z=1$ or $-1/5$. The only two solution candidates are $(x,y,z)=(0,0,1)$ with $\lambda=-1/2$, $\mu=1$, and $(x,y,z)=(4/5,2/5,-1/5)$ with $\lambda=1/2$, $\mu=1/5$. Because $x+y$ is 0 at (0, 0, 1) and 6/5 at $(4/5,2/5,-1/5)$, these are respectively the minimum and the maximum. (The constraints determine geometrically the curve which is the intersection of an ellipsoid (see Fig. 11.4.2) and a plane. The continuous function $x+y$ does attain a maximum and a minimum over this closed bounded set.)

7. (a) With a Cobb–Douglas utility function, $U'_k(\mathbf{x})=\alpha_k U(\mathbf{x})/x_k$, so from (6) (with $j=1$), we have $p_k/p_1=U'_k(\mathbf{x})/U'_1(\mathbf{x})=\alpha_k x_1/\alpha_1 x_k$. Thus $p_k x_k=(a_k/a_1)p_1x_1$. Inserted into the budget constraint, we have $p_1x_1+(a_2/a_1)p_1x_1+\cdots+(a_n/a_1)p_1x_1=m$, which implies that $p_1x_1=a_1m/(a_1+\cdots+a_n)$. Similarly, $p_kx_k=a_km/(a_1+\cdots+a_n)$ for $k=1,\dots,n$.

(b) From (6) (with $j=1$), we get $x_k^{a-1}/x_1^{a-1}=p_k/p_1$ and so $x_k/x_1=(p_k/p_1)^{-1/(1-a)}$ or $p_kx_k/p_1x_1=(p_k/p_1)^{1-1/(1-a)}=(p_k/p_1)^{-a/(1-a)}$. The budget constraint gives

$$p_1x_1\left[1+(p_2/p_1)^{-a/(1-a)}+\cdots+(p_n/p_1)^{-a/(1-a)}\right]=m.\text{ So } p_1x_1=mp_1^{-a/(1-a)}\Big/\sum_{i=1}^n p_i^{-a/(1-a)}.$$

Arguing similarly for each k, one has $x_k=mp_k^{-1/(1-a)}\Big/\sum_{i=1}^n p_i^{-a/(1-a)}$ for $k=1,\dots,n$.

14.7

2. Here $\mathcal{L}=x+4y+3z-\lambda(x^2+2y^2+\frac{1}{3}z^2-b)$. So necessary conditions are:
(i) $\mathcal{L}'_1=1-2\lambda x=0$; (ii) $\mathcal{L}'_2=4-4\lambda y=0$; (iii) $\mathcal{L}'_3=3-\frac{2}{3}\lambda z=0$. It follows that $\lambda\neq 0$, and so $x=1/2\lambda$, $y=1/\lambda$, $z=9/2\lambda$. Inserting these values into the constraint yields $\lambda^2=9/b$, so $\lambda=\pm 3/\sqrt{b}$. The value of the objective function is $x+4y+3z=18/\lambda$, so $\lambda=-3/\sqrt{b}$ determines the minimum point. This is $(x,y,z)=(a,2a,9a)$, where $a=-\sqrt{b}/6$. See the text.

4. With $\mathcal{L}=x^2+y^2+z-\lambda(x^2+2y^2+4z^2-1)$, necessary conditions are: (i) $\partial\mathcal{L}/\partial x=2x-2\lambda x=0$, (ii) $\partial\mathcal{L}/\partial y=2y-4\lambda y=0$, (iii) $\partial\mathcal{L}/\partial z=1-8\lambda z=0$. From (i), $2x(1-\lambda)=0$, so there are two possibilities: $x=0$ or $\lambda=1$.
(A): $x=0$. From (ii), $2y(1-2\lambda)=0$, so $y=0$ or $\lambda=1/2$.
If (A.1), $y=0$, then the constraint gives $4z^2=1$, so $z^2=1/4$, or $z=\pm 1/2$. Equation (iii) gives $\lambda=1/8z$, so we have two solution candidates: $P_1=(0,0,1/2)$ with $\lambda=1/4$ and $P_2=(0,0,-1/2)$ with $\lambda=-1/4$.
(A.2) If $\lambda=1/2$, then (iii) gives $z=1/8\lambda=1/4$. It follows from the constraint that $2y^2=3/4$ (recall that we assumed $x=0$), and hence $y=\pm\sqrt{3/8}=\pm\sqrt{6}/4$. So new candidates are: $P_3=(0,\sqrt{6}/4,1/4)$ with $\lambda=1/2$, $P_4=(0,-\sqrt{6}/4,1/4)$ with $\lambda=1/2$.
(B): Suppose $\lambda=1$. Equation (iii) yields $z=1/8$, and (ii) gives $y=0$. From the constraint, $x^2=15/16$, so $x=\pm\sqrt{15}/4$. Candidates: $P_5=(\sqrt{15}/4,0,1/8)$ with $\lambda=1$, $P_6=(-\sqrt{15}/4,0,1/8)$ with $\lambda=1$.

By computing the values of the criterion function, $f(0,0,1/2)=1/2$, $f(0,0,-1/2)=-1/2$, $f(0,\sqrt{6}/4,1/4)=5/8$, $f(0,-\sqrt{6}/4,1/4)=5/8$, $f(\sqrt{15}/4,0,1/8)=17/16$, $f(-\sqrt{15}/4,0,1/8)=17/16$, we obtain the conclusion in the text. (c) See the text.

5. The Lagrangian is $\mathcal{L}=rK+wL-\lambda(K^{1/2}L^{1/4}-Q)$, so necessary conditions are:
(i) $\mathcal{L}'_K=r-\frac{1}{2}\lambda K^{-1/2}L^{1/4}=0$, (ii) $\mathcal{L}'_L=w-\frac{1}{4}\lambda K^{1/2}L^{-3/4}=0$, (iii) $K^{1/2}L^{1/4}=Q$. From

(i) and (ii) we get by eliminating λ, $L = \frac{1}{2}(rK/w)$. Inserting this into the constraint and solving for K yields the answer given in the text.

14.8

2. (a) Conditions (2)–(3): (i) $2x - 1 - 2\lambda x = 0$; (ii) $4y - 2\lambda y = 0$; (iii) $\lambda \geq 0$, but $\lambda = 0$ if $x^2 + y^2 < 1$.
(b) From (ii), $y(2 - \lambda) = 0$, so either (I) $y = 0$ or (II) $\lambda = 2$.
(I) $y = 0$. If $\lambda = 0$, then from (i), $x = 1/2$ and $(x, y) = (1/2, 0)$ is a candidate for optimum (since it satisfies all the Kuhn–Tucker conditions). If $y = 0$ and $\lambda > 0$, then from (iii) and $x^2 + y^2 \leq 1$, $x^2 + y^2 = 1$. But then $x = \pm 1$, and $(x, y) = (\pm 1, 0)$ are candidates, with $\lambda = 1/2$ and 3/2, respectively.
(II) $\lambda = 2$. Then from (i), $x = -1/2$ and (iii) gives $y^2 = 3/4$, so $y = \pm\sqrt{3}/2$. So $(-1/2, \pm\sqrt{3}/2)$ are the two remaining candidates. For the conclusion, see the text.

3. (a) The Kuhn-Tucker conditions are: (i) $-2(x - 1) - 2\lambda x = 0$; (ii) $-2ye^{y^2} - 2\lambda y = 0$;
(iii) $\lambda \geq 0$, with $\lambda = 0$ if $4x^2 + y^2 < a$. From (i), $x = (1 + \lambda)^{-1}$, and (ii) reduces to $y(e^{y^2} + \lambda) = 0$, and so $y = 0$ (because $e^{y^2} + \lambda$ is always positive).
(I): *Assume that* $\lambda = 0$. Then equation (i) gives $x = 1$. In this case we must have $a \geq x^2 + y^2 = 1$.
(II): *Assume that* $\lambda > 0$. Then (iii) gives $x^2 + y^2 = a$, and so $x = \pm\sqrt{a}$ (remember that $y = 0$). Because $x = 1/(1 + \lambda)$ and $\lambda > 0$ we must have $0 < x < 1$, so $x = \sqrt{a}$ and $a = x^2 < 1$. It remains to find the value of λ and check that it is > 0. From equation (i) we get $\lambda = \dfrac{2(x-1)}{-2x} = \dfrac{1}{x} - 1 = \dfrac{1}{\sqrt{a}} - 1 > 0$.
Conclusion: The only point that satisfies the Kuhn–Tucker conditions is $(x, y) = (1, 0)$ if $a \geq 1$ and $(\sqrt{a}, 0)$ if $0 < a < 1$. The corresponding value of λ is 0 or $\dfrac{1}{\sqrt{a}} - 1$, respectively. In both cases it follows from Theorem 14.8.1 that we have found the maximum point, because $\mathcal{L}$ is concave in (x, y), as we can see by studying the Hessian $\begin{pmatrix} \mathcal{L}''_{11} & \mathcal{L}''_{12} \\ \mathcal{L}''_{21} & \mathcal{L}''_{22} \end{pmatrix} = \begin{pmatrix} -2 - 2\lambda & 0 \\ 0 & -e^{y^2}(2 + 4y^2) - 2\lambda \end{pmatrix}$.
(b) If $a \in (0, 1)$ we have $f^*(a) = f(\sqrt{a}, 0) = 2 - (\sqrt{a} - 1)^2 - 1 = 2\sqrt{a} - a$, and for $a \geq 1$ we get $f^*(a) = f(1, 0) = 1$ (not 2, as it says in the answer section of the first printing of the book). The derivative of f^* is as given in the book, but note that in order to find the derivative $df^*(a)/da$ when $a = 1$, we need to show that the right and left derivatives (see page 242 in the book)

$$(f^*)'(1^+) = \lim_{h\to 0^+} \frac{f^*(1+h) - f^*(1)}{h} \quad \text{and} \quad (f^*)'(1^-) = \lim_{h\to 0^-} \frac{f^*(1+h) - f^*(1)}{h}$$

exist and are equal. The right derivative is obviously 0, since $f^*(a) = 1$ for all $a \geq 1$. To find the left derivative we need to calculate $\lim_{h\to 0^-}(2\sqrt{1+h} - (1 + h) - 1)/h$. A straightforward application of l'Hôpital's rule shows that this limit is also 0. Hence $(f^*)'(1)$ exists and equals 0.

14.9

2. (a) The admissible set is the shaded region in Fig. A14.9.2 in the text.
(b) With the constraints $g_1(x, y) = -x - y \leq -4$, $g_2(x, y) = -x \leq 1$, $g_3(x, y) = -y \leq -1$, the Lagrangian is $\mathcal{L} = x + y - e^x - e^{x+y} - \lambda_1(-x - y + 4) - \lambda_2(-x - 1) - \lambda_3(-y + 1)$. The first-order conditions are that there exist nonnegative numbers λ_1, λ_2, and λ_3 such that (i) $\mathcal{L}'_x = 1 - e^x - e^{x+y} + \lambda_1 + \lambda_2 = 0$; (ii) $\mathcal{L}'_y = 1 - e^{x+y} + \lambda_1 + \lambda_3 = 0$; (iii) $\lambda_1(-x - y + 4) = 0$; (iv) $\lambda_2(-x - 1) = 0$; (v) $\lambda_3(-y + 1) = 0$. (We formulate the complementary slackness conditions as

in (14.8.5).) From (ii), $e^{x+y} = 1 + \lambda_1 + \lambda_3$. Inserting this into (i) yields $\lambda_2 = e^x + \lambda_3 \geq e^x > 0$. Because $\lambda_2 > 0$, (iv) implies that $x = -1$. So any solution must lie on the line (II) in the figure, which shows that the third constraint must be slack. (Algebraically, because $x + y \geq 4$ and $x = -1$, we have $y \geq 4 - x = 5 > 1$.) So from (v) we get $\lambda_3 = 0$, and then (ii) gives $\lambda_1 = e^{x+y} - 1 \geq e^4 - 1 > 0$. Thus from (iii), the first constraint is active, so $y = 4 - x = 5$. Hence the only possible solution is $(x^*, y^*) = (-1, 5)$. Because $\mathcal{L}(x, y)$ is concave, we have found the optimal point.

3. (a) The feasible set is shown in Fig. A14.9.3 in the book. (The function to be maximized is $f(x, y) = x + ay$. The level curves of this function are straight lines with slope $-1/a$ if $a \neq 0$, and vertical lines if $a = 0$. The dashed line in the figure is such a level curve (for $a \approx -0.25$). The maximum point for f is that point in the feasible region that we shall find if we make a parallel displacement of this line as far to the right as possible (why to the right?) without losing contact with the shaded region.)

The Lagrangian is $\mathcal{L}(x, y) = x + ay - \lambda_1(x^2 + y^2 - 1) + \lambda_2(x + y)$ (the second constraint must be written as $-x - y \leq 0$), so the Kuhn–Tucker conditions are:
(i) $\mathcal{L}_1'(x, y) = 1 - 2\lambda_1 x + \lambda_2 = 0$; (ii) $\mathcal{L}_2'(x, y) = a - 2\lambda_1 y + \lambda_2 = 0$;
(iii) $\lambda_1 \geq 0$, with $\lambda_1 = 0$ if $x^2 + y^2 < 1$; (iv) $\lambda_2 \geq 0$, with $\lambda_2 = 0$ if $x + y > 0$.

(b) From (i), $2\lambda_1 x = 1 + \lambda_2 \geq 1 > 0$, so since from (iii) $\lambda_1 \geq 0$, we must have $\lambda_1 > 0$ and also $x > 0$. Because $\lambda_1 > 0$, it follows from (iii) that $x^2 + y^2 = 1$, so any maximum point must lie on the circle.
(I) First assume that $x + y = 0$. Then $y = -x$, and since $x^2 + y^2 = 1$, we get $x = \frac{1}{2}\sqrt{2}$ (recall that we have seen that x must be positive) and $y = -\frac{1}{2}\sqrt{2}$. Adding equations (i) and (ii), we get

$$1 + a - 2\lambda_1(x + y) + 2\lambda_2 = 0$$

and since $x + y = 0$, we find that $\lambda_2 = -(1 + a)/2$. Now, λ_2 must be ≥ 0, and therefore we must have $a \leq -1$ in this case. Equation (i) gives $\lambda_1 = 1 + \lambda_2/2x = 1 - a/4x = 1 - a/2\sqrt{2}$.
(II) Then consider the case $x + y > 0$. Then $\lambda_2 = 0$, and we get $1 - 2\lambda_1 x = 0$ and $a - 2\lambda_1 y = 0$, which gives $x = 1/(2\lambda_1)$ and $y = a/(2\lambda_1)$. Since (x, y) must lie on the circle, we then get $1 = x^2 + y^2 = 1 + a^2/4\lambda_1^2$, and therefore $2\lambda_1 = \sqrt{1 + a^2}$. This gives $x = \dfrac{1}{\sqrt{1 + a^2}}$ and $y = \dfrac{a}{\sqrt{1 + a^2}}$. Because $x + y = (1 + a)/(2\lambda_1)$, and because $x + y$ is now assumed to be positive, we must have $a > -1$ in this case. *Conclusion:* The only points satisfying the Kuhn–Tucker conditions are the ones given in the text. Since the feasible set is closed and bounded and f is continuous, it follows from the extreme value theorem that extreme points exists.

4. (a) The Lagrangian is $\mathcal{L} = y - x^2 + \lambda y + \mu(y - x + 2) - \nu(y^2 - x)$, which is stationary when (i) $-2x - \mu + \nu = 0$; (ii) $1 + \lambda + \mu - 2\nu y = 0$. Complementary slackness requires in addition, (iii) $\lambda \geq 0$, with $\lambda = 0$ if $y > 0$; (iv) $\mu \geq 0$, with $\mu = 0$ if $y - x > -2$; (v) $\nu \geq 0$, with $\nu = 0$ if $y^2 < x$.

From (ii), $2\nu y = 1 + \lambda + \mu > 0$, so $y > 0$. Then (iii) implies $\lambda = 0$, and $2\nu y = 1 + \mu$. From (i), $x = \frac{1}{2}(\nu - \mu)$. But $x \geq y^2 > 0$, so $\nu > \mu \geq 0$, and from (v), $y^2 = x$.

Suppose $\mu > 0$. Then $y - x + 2 = y - y^2 + 2 = 0$ with roots $y = -1$ and $y = 2$. Only $y = 2$ is feasible. Then $x = y^2 = 4$. Because $\lambda = 0$, conditions (i) and (ii) become $-\mu + \nu = 8$ and $\mu - 4\nu = -1$, so $\nu = -7/3$, which contradicts $\nu \geq 0$, so $(x, y) = (4, 2)$ is not a candidate. Therefore $\mu = 0$ after all. Thus $x = \frac{1}{2}\nu = y^2$ and, by (ii), $1 = 2\nu y = 4y^3$. Hence $y = 4^{-1/3}$, $x = 4^{-2/3}$. This is the only remaining candidate. It is the solution with $\lambda = 0$, $\mu = 0$, and $\nu = 1/2y = 4^{-1/6}$.

(b) We write the problem as $\max xe^{y-x} - 2ey$ subject to $y \leq 1 + \frac{1}{2}x$, $x \geq 0$, $y \geq 0$. The Lagrangian is $\mathcal{L} = xe^{y-x} - 2ey - \lambda(y - 1 - x/2)$, so the first-order conditions (14.9.4) and (14.9.5) are:

(i) $\mathcal{L}'_x = e^{y-x} - xe^{y-x} + \frac{1}{2}\lambda \le 0$ ($= 0$ if $x > 0$); (ii) $\mathcal{L}'_y = xe^{y-x} - 2e - \lambda \le 0$ ($= 0$ if $y > 0$); (iii) $\lambda \ge 0$ with $\lambda = 0$ if $y < 1 + \frac{1}{2}x$.

From (i) we have $x \ge 1 + \frac{1}{2}\lambda e^{x-y} \ge 1$, so (iv) $(x-1)e^{y-x} = \frac{1}{2}\lambda$. Suppose $\lambda > 0$. Then (iii) implies (v) $y = 1 + \frac{1}{2}x > 0$. From (ii) and (iv) we then have $xe^{y-x} = 2e + \lambda = e^{y-x} + \frac{1}{2}\lambda$ and so $\lambda = 2e^{y-x} - 4e = 2e(e^{-\frac{1}{2}x} - 2)$, by (v). But then $\lambda > 0$ implies that $e^{-\frac{1}{2}x} > 2$, which contradicts $x \ge 0$. When $\lambda = 0$, (iv) gives $x = 1$. If $y > 0$, then (ii) yields $e^{y-1} = 2e$, and so $y - 1 = \ln(2e) > \frac{1}{2}x$ when $x = 1$. Thus feasibility requires that $y = 0$, so we see that $(x, y) = (1, 0)$ is the only point satisfying all the conditions, with $\lambda = 0$.

5. A feasible triple (x_1^*, x_2^*, k^*) solves the problem iff there exist numbers λ and μ such that (i) $1 - 2x_1^* - \lambda \le 0$ ($= 0$ if $x_1^* > 0$); (ii) $3 - 2x_2^* - \mu \le 0$ ($= 0$ if $x_2^* > 0$); (iii) $-2k^* + \lambda + \mu \le 0$ ($= 0$ if $k^* > 0$); (iv) $\lambda \ge 0$ with $\lambda = 0$ if $x_1^* < k^*$; and (v) $\mu \ge 0$ with $\mu = 0$ if $x_2^* < k^*$.

If $k^* = 0$, then feasibility requires $x_1^* = 0$ and $x_2^* = 0$, and so (i) and (iii) imply that $\lambda \ge 1$ and $\mu \ge 3$, which contradicts (iii). Thus, $k^* > 0$. Next, if $\mu = 0$, then $x_2^* \ge 3/2$ and $\lambda = 2k^* > 0$. So $x_1^* = k^* = 1/4$, contradicting $x_2^* \le k^*$. So $\mu > 0$, which implies that $x_2^* = k^*$. Now, if $x_1^* = 0 < k^*$, then $\lambda = 0$, which contradicts (i). So $0 < x_1^* = \frac{1}{2}(1-\lambda)$. Next, if $\lambda > 0$, then $x_1^* = k^* = x_2^* = \frac{1}{2}(1-\lambda) = \frac{1}{2}(3-\mu) = \frac{1}{2}(\lambda+\mu)$. But the last two equalities are only satisfied when $\lambda = -1/3$ and $\mu = 5/3$, which contradicts $\lambda \ge 0$. So $\lambda = 0$ after all, with $x_2^* = k^* > 0$, $\mu > 0$, $x_1^* = \frac{1}{2}(1-\lambda) = \frac{1}{2}$. Now, from (iii) it follows that $\mu = 2k^*$ and so, from (ii), that $3 = 2x_2^* + \mu = 4k^*$. The only possible solution is, therefore, $(x_1^*, x_2^*, k^*) = (1/2, 3/4, 3/4)$, with $\lambda = 0$ and $\mu = 3/2$. (The Lagrangian *is* concave in (x_1, x_2, k). See FMEA, Theorem 3.2.4.)

6. A minus sign has disappeared in the objective function which should be: $-(x + \frac{1}{2})^2 - \frac{1}{2}y^2$.
(a) See Fig. A14.9.6 in the text. Note that for (x, y) to be admissible, $e^{-x} \le y \le 2/3$, and so $e^x \ge 3/2$.
(b) The Lagrangian is $\mathcal{L} = -(x+\frac{1}{2})^2 - \frac{1}{2}y^2 - \lambda_1(e^{-x} - y) - \lambda_2(y - \frac{2}{3})$, and the first-order conditions are: (i) $-(2x+1) + \lambda_1 e^{-x} = 0$; (ii) $-y + \lambda_1 - \lambda_2 = 0$; (iii) $\lambda_1 \ge 0$, with $\lambda_1 = 0$ if $e^{-x} < y$; (iv) $\lambda_2 \ge 0$, with $\lambda_2 = 0$ if $y < 2/3$. From (i), $\lambda_1 = (2x+1)e^x \ge 3/2$, because of (a). From (ii), $\lambda_2 = \lambda_1 - y \ge 3/2 - 2/3 > 0$, so $y = 2/3$ because of (iii). Solution: $(x^*, y^*) = (\ln(3/2), 2/3)$, with $\lambda_1 = 3[\ln(3/2) + 1/2]$, $\lambda_2 = 3\ln(3/2) + 5/6$. The Lagrangian is concave so this is the solution.

Alternative argument: Suppose $\lambda_1 = 0$. Then from (ii), $y = -\lambda_2 \le 0$, contradicting $y \ge e^{-x}$. So $\lambda_1 > 0$, and (iii) gives $y = e^{-x}$. Suppose $\lambda_2 = 0$. Then from (ii), $\lambda_1 = y = e^{-x}$ and (i) gives $e^{-2x} = 2x + 1$. Define $g(x) = 2x + 1 - e^{-2x}$. Then $g(0) = 0$ and $g'(x) = 2 + 2e^{-2x} > 0$. So the equation $e^{-2x} = 2x + 1$ has no solution except $x = 0$. Thus $\lambda_2 > 0$, etc.

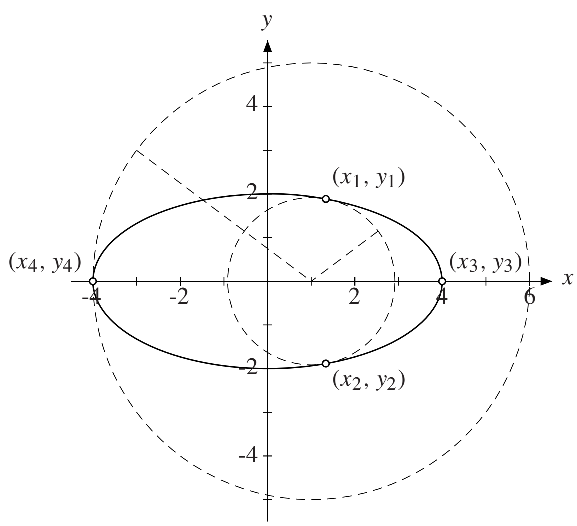
Review Problems for Chapter 14

3. (a) Interpretation of the first-order condition $p(x^*) = C_1'(x^*, y^*) - x^* p'(x^*)$: How much is gained by selling one ton extra of the first commodity? $p(x^*)$, because this is the price obtained for one ton. How much is lost? First, selling one ton extra of the first commodity incurs the extra cost $C(x^*+1, y^*) - C(x^*, y^*)$, which is approximately $C_1'(x^*, y^*)$. But since presumably $p'(x) < 0$, producing one ton extra leads to a decrease in income which is approximately $-x^* p'(x^*)$ (the number of tons sold times the decrease in the price. So what we loose by increasing production by 1 ton $(C_1'(x^*, y^*) - x^* p'(x^*))$ is approximately what we gain $(p(x^*))$. The other first-order condition, $q(y^*) = C_2'(x^*, y^*) - y^* q'(y^*)$ has a similiar interpretation.
(b) See the text. If we assume that the restriction is $x + y \le m$, we have to add the condition $\lambda \ge 0$, with $\lambda = 0$ if $\hat{x} + \hat{y} < m$.

4. See the text. If one were to find the partial derivatives of x and y w.r.t. p as well, it would be better to calculate differentials, which would yield the equations (i) $y\,dp + p\,dy = 24\,dw - w\,dx - x\,dw$ and (ii) $U_1'\,dp + p(U_{11}''\,dx + U_{12}''\,dy) = U_2'\,dw + wU_{21}''\,dx + wU_{22}''\,dy$, and then solve these equations for dx and dy in terms of dp and dw.

5. (a) With $\mathcal{L} = x^2 + y^2 - 2x + 1 - \lambda(\frac{1}{4}x^2 + y^2 - b)$, the first-order conditions are:
(i) $\mathcal{L}_1' = 2x - 2 - \frac{1}{2}\lambda x = 0$; (ii) $\mathcal{L}_2' = 2y - 2\lambda y = 0$; (iii) $\frac{1}{4}x^2 + y^2 = b$.
From (ii), $(1-\lambda)y = 0$, and thus $\lambda = 1$ or $y = 0$.
(I) Suppose first that $\lambda = 1$. Then (i) gives $x = \frac{4}{3}$, and from (iii) we have $y^2 = b - \frac{1}{4}x^2 = b - \frac{4}{9}$, which gives $y = \pm\sqrt{b - \frac{4}{9}}$. This gives two candidates: $(x_1, y_1) = (4/3, \sqrt{b - \frac{4}{9}})$ and $(x_2, y_2) = (4/3, -\sqrt{b - \frac{4}{9}})$.
(II) If $y = 0$, then from (iii), $x^2 = 4b$, i.e. $x = \pm 2\sqrt{b}$. This gives two further candidates: $(x_3, y_3) = (2\sqrt{b}, 0)$ and $(x_4, y_4) = (-2\sqrt{b}, 0)$. The objective function evaluated at the candidates are: $f(x_1, y_1) = f(x_2, y_2) = b - 1/3$, $f(x_3, y_3) = (2\sqrt{b} - 1)^2 = 4b - 4\sqrt{b} + 1$, $f(x_4, y_4) = (-2\sqrt{b} - 1)^2 = 4b + 4\sqrt{b} + 1$
Clearly, (x_4, y_4) is the maximum point. To decide which of the points (x_3, y_3), (x_1, y_1), or (x_2, y_2) give the minimum, we have to decide which of $4b - 4\sqrt{b} + 1$ and $b - \frac{1}{3}$ is the largest. The difference is $4b - 4\sqrt{b} + 1 - \big(b - \frac{1}{3}\big) = 3\big(b - \frac{4}{3}\sqrt{b} + \frac{4}{9}\big) = 3\big(\sqrt{b} - \frac{2}{3}\big)^2 > 0$ since $b > \frac{4}{9}$. Thus the minimum occurs at (x_1, y_1) and (x_2, y_2).

The constraint $x^2/4 + y^2 = b$ is the ellipse indicated in the figure. The objective function $f(x, y) = (x-1)^2 + y^2$ is the square of the distance between (x, y) and the point $(1, 0)$. The level curves for f are therefore circles centred at $(1, 0)$, and in the figure we see those two that passes through the maximum and minimum points. (b) See the text.



For Problem 14.R.5

7. (a) With $\mathcal{L} = x^2 - 2x + 1 + y^2 - 2y - \lambda[(x+y)\sqrt{x+y+b} - 2\sqrt{a}]$, the first-order conditions are:
(i) $\mathcal{L}_1' = 2x - 2 - \lambda[\sqrt{x+y+b} + (x+y)/\sqrt{x+y+b}\,] = 0$,
(ii) $\mathcal{L}_2' = 2y - 2 - \lambda[\sqrt{x+y+b} + (x+y)/\sqrt{x+y+b}\,] = 0$. From these equations it follows immediately that $2x - 2 = 2y - 2$, so $x = y$. See the text.

(b) Differentiation yields: (i) $dx = dy$; (ii) $6x^2\,dx + x^2\,db + 2bx\,dx = da$. From these equations we easily read off the first-order partials of x and y w.r.t. a and b. Further,

$$\frac{\partial^2 x}{\partial a^2} = \frac{\partial}{\partial a}\left(\frac{\partial x}{\partial a}\right) = \frac{\partial}{\partial a}\frac{1}{6x^2+2bx} = -\frac{12x+2b}{(6x^2+2bx)^2}\frac{\partial x}{\partial a} = -\frac{12x+2b}{(6x^2+2bx)^3} = -\frac{6x+b}{4(3x^2+bx)^3}$$

8. (a) The Lagrangian is $\mathcal{L} = xy - \lambda_1(x^2+ry^2-m) - \lambda_2(-x)$, and the necessary Kuhn–Tucker conditions for (x^*, y^*) to solve the problem are:

(i) $\mathcal{L}_1' = y^* - 2\lambda_1 x^* + \lambda_2 = 0$;
(ii) $\mathcal{L}_2' = x^* - 2r\lambda_1 y^* = 0$;
(iii) $\lambda_1 \geq 0$, with $\lambda_1 = 0$ if $(x^*)^2 + r(y^*)^2 < m$;
(iv) $\lambda_2 \geq 0$, with $\lambda_2 = 0$ if $x^* > 1$;
(v) $(x^*)^2 + r(y^*)^2 \leq m$;
(vi) $x^* \geq 1$

(b) From (ii) and (vi) we see that $\lambda_1 = 0$ is impossible. Thus $\lambda_1 > 0$, and from (iii) and (v),

(vii) $(x^*)^2 + r(y^*)^2 = m$.

(I): Assume $\lambda_2 = 0$. Then from (i) and (ii), $y^* = 2\lambda_1 x^*$ and $x^* = 2\lambda_1 r y^*$, so $y^* = 4\lambda_1^2 r y^*$. If $y^* = 0$, then (ii) implies $x^* = 0$, which is impossible. Hence, $\lambda_1^2 = 1/4r$ and thus $\lambda_1 = 1/2\sqrt{r}$. Then $y^* = x^*/\sqrt{r}$, which inserted into (vii) and solved for x^* yields $x^* = \sqrt{m/2}$ and then $y^* = \sqrt{m/2r}$. Note that $x^* \geq 1 \iff \sqrt{m/2} \geq 1 \iff m \geq 2$. Thus for $m \geq 2$, $x^* = \sqrt{m/2}$ and $y^* = \sqrt{m/2r}$, with $\lambda_1 = 1/2\sqrt{r}$ and $\lambda_2 = 0$ is a solution candidate.
(II): Assume $\lambda_2 > 0$. Then $x^* = 1$ and from (vii) we have $r(y^*)^2 = m - 1$, so $y^* = \sqrt{(m-1)/r}$ ($y^* = -\sqrt{(m-1)/r}$ contradicts (ii)). Inserting these values for x^* and y^* into (i) and (ii) and solving for λ_1 and λ_2 yields $\lambda_1 = 1/2\sqrt{r(m-1)}$ and furthermore, $\lambda_2 = (2-m)/\sqrt{r(m-1)}$. Note that $\lambda_2 > 0 \iff 1 < m < 2$. Thus, for $1 < m < 2$, the only solution candidate is $x^* = 1$, $y^* = \sqrt{(m-1)/r}$, with $\lambda_1 = 1/2\sqrt{r(m-1)}$ and $\lambda_2 = (2-m)/\sqrt{r(m-1)}$.

The objective function is continuous and the constraint set is obviously closed and bounded, so by the extreme value theorem there has to be a maximum. The solution candidates we have found are therefore optimal. (Alternatively, $\mathcal{L}_{11}'' = -2\lambda_1 \leq 0$, $\mathcal{L}_{22}'' = -2r\lambda_1 \leq 0$, and $\Delta = \mathcal{L}_{11}''\mathcal{L}_{22}'' - (\mathcal{L}_{12}'')^2 = 4r\lambda_1^2 - 1$. In the case $m \geq 2$, $\Delta = 0$, and in the case $1 < m < 2$, $\Delta = 1/(m-1) > 0$. Thus in both cases, $\mathcal{L}(x, y)$ is concave.)

15 Matrix and Vector Algebra

15.1

2. Here is one method: Adding the 4 equations gives $x_1+x_2+x_3+x_4 = \frac{1}{3}(b_1+b_2+b_3+b_4)$, after dividing by 3. From this equation, subtracting each original equation in turn gives $x_1 = -\frac{2}{3}b_1 + \frac{1}{3}(b_2+b_3+b_4)$, $x_2 = -\frac{2}{3}b_2 + \frac{1}{3}(b_1+b_3+b_4)$, $x_3 = -\frac{2}{3}b_3 + \frac{1}{3}(b_1+b_2+b_4)$, $x_4 = -\frac{2}{3}b_4 + \frac{1}{3}(b_1+b_2+b_3)$.

Systematic elimination of the variables starting by eliminating (say) x_4 is an alternative.

6. Suggestion: Solve the first equation for y. Insert this expression for y and $x = 93.53$ into the third equation. Solve it for s. Insert the results into the second equation and solve for c, etc.

15.3

1. (a) $\begin{pmatrix} 0 & -2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} -1 & 4 \\ 1 & 5 \end{pmatrix} = \begin{pmatrix} 0\cdot(-1)+(-2)\cdot 1 & 0\cdot 4+(-2)\cdot 5 \\ 3\cdot(-1)+\quad 1\cdot 1 & 3\cdot 4+1\cdot 5 \end{pmatrix} = \begin{pmatrix} -2 & -10 \\ -2 & 17 \end{pmatrix}$

The rest goes in the same way. Note that in (d) **AB** is not defined because the number of columns in **A** is not equal to the number of rows in **B**.

5. (a) We know that **A** is an $m \times n$ matrix. Let **B** be a $p \times q$ matrix. The matrix product **AB** is defined if and only if $n = p$, and **BA** is defined if and only if $q = m$. So for both **AB** and **BA** to be defined, it is necessary and sufficient that **B** is an $n \times m$ matrix.
(b) We know from part (a) that if **BA** and **AB** are defined, then **B** must be a 2×2 matrix. So let $\mathbf{B} = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$. Then $\mathbf{BA} = \begin{pmatrix} x & y \\ z & w \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} x+2y & 2x+3y \\ z+2w & 2z+3w \end{pmatrix}$, and $\mathbf{AB} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x+2z & y+2w \\ 2x+3z & 2y+3w \end{pmatrix}$. Hence, $\mathbf{BA} = \mathbf{AB}$ iff (i) $x + 2y = x + 2z$, (ii) $2x + 3y = y + 2w$, (iii) $z + 2w = 2x + 3z$, and (iv) $2z + 3w = 2y + 3w$. The first and last of these four equations are true if and only if $y = z$, and if $y = z$, then the second and third are true if and only if $x = w - y$. Hence, the matrices **B** that commute with **A** are precisely the matrices of the form

$$\mathbf{B} = \begin{pmatrix} w-y & y \\ y & w \end{pmatrix} = w\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + y\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$$

where y and w can be any real numbers.

15.4

2. We start by performing the multiplication $\begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ax+dy+ez \\ dx+by+fz \\ ex+fy+cz \end{pmatrix}$. Next,
$(x,\ y,\ z)\begin{pmatrix} ax+dy+ez \\ dx+by+fz \\ ex+fy+cz \end{pmatrix} = (ax^2+by^2+cz^2+2dxy+2exz+2fyz)$, which is a 1×1 matrix.

7. (a) Direct verification yields (i) $\mathbf{A}^2 = (a+d)\mathbf{A} - (ad-bc)\mathbf{I}_2 = \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix}$

(b) For the matrix **A** in (a), $\mathbf{A}^2 = \mathbf{0}$ if $a + d = 0$ and $ad = bc$, so one example with $\mathbf{A} \neq \mathbf{0}$ is $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$.

(c) Multiplying (i) in (a) by **A** and using $\mathbf{A}^3 = \mathbf{0}$ yields (ii) $(a+d)\mathbf{A}^2 = (ad-bc)\mathbf{A}$. Multiplying by **A** once more gives $(ad-bc)\mathbf{A}^2 = \mathbf{0}$. If $ad - bc \neq 0$, then $\mathbf{A}^2 = \mathbf{0}$. If $ad - bc = 0$, (ii) yields $(a+d)\mathbf{A}^2 = \mathbf{0}$, and if $a + d \neq 0$, again $\mathbf{A}^2 = \mathbf{0}$. Finally, if $ad - bc = a + d = 0$, then (i) implies $\mathbf{A}^2 = \mathbf{0}$.

15.5

6. In general, for any natural number $n > 3$, one has $((\mathbf{A}_1\mathbf{A}_2\cdots\mathbf{A}_{n-1})\mathbf{A}_n)' = \mathbf{A}_n'(\mathbf{A}_1\mathbf{A}_2\cdots\mathbf{A}_{n-1})'$. As the induction hypothesis, suppose the result is true for $n - 1$. Then the last expression becomes $\mathbf{A}_n'\mathbf{A}_{n-1}'\cdots,\mathbf{A}_2'\mathbf{A}_1'$, so the result is true for n.

8. (a) $\mathbf{TS} = \mathbf{S}$ is shown in the text. A similar argument shows that $\mathbf{T}^2 = \frac{1}{2}\mathbf{T} + \frac{1}{2}\mathbf{S}$. To prove the last equality, we do not have to consider the individual elements: $\mathbf{T}^3 = \mathbf{TT}^2 = \mathbf{T}(\frac{1}{2}\mathbf{T}+\frac{1}{2}\mathbf{S}) = \frac{1}{2}\mathbf{T}^2+\frac{1}{2}\mathbf{TS} = \frac{1}{2}(\frac{1}{2}\mathbf{T}+\frac{1}{2}\mathbf{S})+\frac{1}{2}\mathbf{S} = \frac{1}{4}\mathbf{T}+\frac{3}{4}\mathbf{S}$.

(b) The appropriate formula is $(*)$ $\mathbf{T}^n = 2^{1-n}\mathbf{T} + (1-2^{1-n})\mathbf{S}$. This formula is correct for $n = 1$ (and for $n = 2, 3$). Suppose $(*)$ is true for $n = k$. Then using the two first equalities in (a), $\mathbf{T}^{k+1} = \mathbf{TT}^k = \mathbf{T}(2^{1-k}\mathbf{T} + (1-2^{1-k})\mathbf{S}) = 2^{1-k}\mathbf{T}^2 + (1-2^{1-k})\mathbf{TS} = 2^{1-k}(\frac{1}{2}\mathbf{T}+\frac{1}{2}\mathbf{S}) + (1-2^{1-k})\mathbf{S} = 2^{-k}\mathbf{T} + 2^{-k}\mathbf{S} + \mathbf{S} - 2\cdot 2^{-k}\mathbf{S} = 2^{-k}\mathbf{T} + (1-2^{-k})\mathbf{S}$, which is formula $(*)$ for $n = k+1$.

15.6

3. By using elementary operations, we find that

$$\begin{matrix} w & x & y & z & \\ \end{matrix}$$
$$\begin{pmatrix} 2 & 1 & 4 & 3 & 1 \\ 1 & 3 & 2 & -1 & 3c \\ 1 & 1 & 2 & 1 & c^2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 & 2 & 1-c^2 \\ 0 & 1 & 0 & -1 & 2c^2-1 \\ 0 & 0 & 0 & 0 & -5c^2+3c+2 \end{pmatrix}$$

We can tell from the last matrix that the system has solutions if and only if $-5c^2+3c+2 = 0$, that is, if and only if $c = 1$ or $c = -2/5$. For these particular values of c we get the solutions in the text.

4. (a) After moving the first row down to row number three, Gaussian elimination yields the matrix $\begin{pmatrix} 1 & 2 & 1 & b_2 \\ 0 & 1 & -2 & \frac{3}{2}b_2 - \frac{1}{2}b_3 \\ 0 & 0 & 3-4a & b_1 + (2a-\frac{3}{2})b_2 + (\frac{1}{2}-a)b_3 \end{pmatrix}$. Obviously, there is a unique solution iff $a \neq 3/4$.

(b) Put $a = 3/4$ in part (a). Then the last row in the matrix in (a) becomes $(0, 0, 0, b_1 - \frac{1}{4}b_3)$. It follows that if $b_1 \neq \frac{1}{4}b_3$ there is no solution. If $b_1 = \frac{1}{4}b_3$ there are an infinite number of solutions. In fact, $x = -2b_2 + b_3 - 5t$, $y = \frac{3}{2}b_2 - \frac{1}{2}b_3 + 2t$, $z = t$, with $t \in \mathbb{R}$.

15.7

3. Using the definitions of vector addition and multiplication of a vector by a real number, we get $3(x, y, z) + 5(-1, 2, 3) = (4, 1, 3) \iff (3x-5, 3y+10, 3z+15) = (4, 1, 3)$. Since two vectors are equal if and only if they are component-wise equal, this vector equation is equivalent to the equation system $3x - 5 = 4$, $3y + 10 = 1$, and $3z + 15 = 3$, with the obvious solution $x = 3$, $y = -3$, $z = -4$.

5. We need to find numbers t and s such that $t(2, -1)+s(1, 4) = (4, -11)$. This vector equation is equivalent to $(2t + s, -t + 4s) = (4, -11)$, which in turn is equivalent to the equation system (i) $2t + s = 4$ (ii) $-t + 4s = -11$. This system has the solution $t = 3$, $s = -2$, so $(4, -11) = 3(2, -1) - 2(1, 4)$.

15.8

2. (a) See the text. (b) As λ runs through $[0, 1]$, the vector $\mathbf{x}$ will run through all points on the line segment S between $\mathbf{a}$ and $\mathbf{b}$. In fact, according to the point–point formula, the line L through $(3, 1)$ and $(-1, 2)$ has the equation $x_2 = -\frac{1}{4}x_1 + \frac{7}{4}$ or $x_1 + 4x_2 = 7$. The line segment S is traced out by having x_1 run through $[3, -1]$ as x_2 runs through $[1, 2]$. Now, $(1-\lambda)\mathbf{a} + \lambda\mathbf{b} = (3-4\lambda, 1+\lambda)$. Any point (x_1, x_2) on L satisfies $x_1 + 4x_2 = 7$ and equals $(3-4\lambda, 1+\lambda)$ for $\lambda = \frac{1}{4}(3-x_1) = x_2 - 1$. Any point on the segment of this line between $\mathbf{a} = (3, 1)$ and $\mathbf{b} = (-1, 2)$ equals $(3-4\lambda, 1+\lambda)$ for some $\lambda \in [0, 1]$.

8. $(\|\mathbf{a}\| + \|\mathbf{b}\|)^2 - \|\mathbf{a}+\mathbf{b}\|^2 = \|\mathbf{a}\|^2 + 2\|\mathbf{a}\|\cdot\|\mathbf{b}\| + \|\mathbf{b}\|^2 - (\mathbf{a}+\mathbf{b})(\mathbf{a}+\mathbf{b}) = 2(\|\mathbf{a}\|\cdot\|\mathbf{b}\| - \mathbf{a}\cdot\mathbf{b}) \geq 2(\|\mathbf{a}\|\cdot\|\mathbf{b}\| - |\mathbf{a}\cdot\mathbf{b}|) \geq 0$ by the Cauchy–Schwarz inequality (2).

15.9

3. One method: $(5, 2, 1) - (3, 4, -3) = (2, -2, 4)$ and $(2, -1, 4) - (3, 4, -3) = (-1, -5, 7)$ are two vectors in the plane. The normal (p_1, p_2, p_3) must be orthogonal to both these vectors, so $(2, -2, 4)\cdot(p_1, p_2, p_3) = 2p_1 - 2p_2 + 4p_3 = 0$ and $(-1, -5, 7)\cdot(p_1, p_2, p_3) = -p_1 - 5p_2 + 7p_3 = 0$. One solution of these two equations is $(p_1, p_2, p_3) = (1, -3, -2)$. Then using formula (4) with $(a_1, a_2, a_3) = (2, -1, 4)$ yields $(1, -3, -2)\cdot(x_1 - 2, x_2 + 1, x_3 - 4) = 0$, or $x_1 - 3x_2 - 2x_3 = -3$.

A more pedestrian approach is to assume that the equation is $ax + by + cz = d$ and require the three points to satisfy the equation: $a + 2c = d$, $5a + 2b + c = d$, $2a - b + 4c = d$. Solve for a, b, and c in terms of d, insert the results into the equation $ax + by + cz = d$ and cancel d.

Review Problems for Chapter 15

7. (a) $\begin{pmatrix} 1 & 4 & 1 \\ 2 & 2 & 8 \end{pmatrix} \begin{matrix} -2 \\ \leftarrow \end{matrix} \sim \begin{pmatrix} 1 & 4 & 1 \\ 0 & -6 & 6 \end{pmatrix} \begin{matrix} \\ -1/6 \end{matrix} \sim \begin{pmatrix} 1 & 4 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{matrix} \leftarrow \\ -4 \end{matrix} \sim \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & -1 \end{pmatrix}$

The solution is $x_1 = 5$, $x_2 = -1$.

(b) $\begin{pmatrix} 2 & 2 & -1 & 2 \\ 1 & -3 & 1 & 0 \\ 3 & 4 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -3 & 1 & 0 \\ 2 & 2 & -1 & 2 \\ 3 & 4 & -1 & 1 \end{pmatrix} \begin{matrix} -2 \quad -3 \\ \leftarrow \\ \leftarrow \end{matrix} \sim \begin{pmatrix} 1 & -3 & 1 & 0 \\ 0 & 8 & -3 & 2 \\ 0 & 13 & -4 & 1 \end{pmatrix} \begin{matrix} \\ 1/8 \\ \\ \end{matrix}$

$\sim \begin{pmatrix} 1 & -3 & 1 & 0 \\ 0 & 1 & -3/8 & 1/4 \\ 0 & 13 & -4 & 1 \end{pmatrix} \begin{matrix} \\ -13 \\ \leftarrow \end{matrix} \sim \begin{pmatrix} 1 & -3 & 1 & 0 \\ 0 & 1 & -3/8 & 1/4 \\ 0 & 0 & 7/8 & -9/4 \end{pmatrix} \begin{matrix} \\ \\ 8/7 \end{matrix}$

$\sim \begin{pmatrix} 1 & -3 & 1 & 0 \\ 0 & 1 & -3/8 & 1/4 \\ 0 & 0 & 1 & -18/7 \end{pmatrix} \begin{matrix} \leftarrow \\ 3 \\ \\ \end{matrix} \sim \begin{pmatrix} 1 & 0 & -1/8 & 3/4 \\ 0 & 1 & -3/8 & 1/4 \\ 0 & 0 & 1 & -18/7 \end{pmatrix} \begin{matrix} \leftarrow \\ \leftarrow \\ 3/8 \quad 1/8 \end{matrix}$

$\sim \begin{pmatrix} 1 & 0 & 0 & 3/7 \\ 0 & 1 & 0 & -5/7 \\ 0 & 0 & 1 & -18/7 \end{pmatrix}$. The solution is $x_1 = 3/7$, $x_2 = -5/7$, $x_3 = -18/7$.

(c) $\begin{pmatrix} 1 & 3 & 4 & 0 \\ 5 & 1 & 1 & 0 \end{pmatrix} \begin{matrix} -5 \\ \leftarrow \end{matrix} \sim \begin{pmatrix} 1 & 3 & 4 & 0 \\ 0 & -14 & -19 & 0 \end{pmatrix} \begin{matrix} \\ -1/14 \end{matrix}$

$\sim \begin{pmatrix} 1 & 3 & 4 & 0 \\ 0 & 1 & 19/14 & 0 \end{pmatrix} \begin{matrix} \leftarrow \\ -3 \end{matrix} \sim \begin{pmatrix} 1 & 0 & -1/14 & 0 \\ 0 & 1 & 19/14 & 0 \end{pmatrix}$

The solution is $x_1 = (1/14)x_3$, $x_2 = -(19/14)x_3$, where x_3 is arbitrary. (One degree of freedom.)

10. (a) See the text.

(b) In (a) we saw that **a** can be produced even without throwing away outputs. For **b** to be possible if we are allowed to throw away output, there must exist a λ in $[0, 1]$ such that $6\lambda + 2 \geq 7$, $-2\lambda + 6 \geq 5$, and $-6\lambda + 10 \geq 5$. These inequalities reduce to $\lambda \geq 5/6$, $\lambda \leq 1/2$, $\lambda \leq 5/6$, which are incompatible.

(c) Revenue $= R(\lambda) = p_1x_1 + p_2x_2 + p_3x_3 = (6p_1 - 2p_2 - 6p_3)\lambda + 2p_1 + 6p_2 + 10p_3$. If the constant slope $6p_1 - 2p_2 - 6p_3$ is > 0, then $R(\lambda)$ is maximized at $\lambda = 1$; if $6p_1 - 2p_2 - 6p_3$ is < 0, then $R(\lambda)$ is maximized at $\lambda = 0$. Only in the special case where $6p_1 - 2p_2 - 6p_3 = 0$ can the two plants both remain in use.

11. If $\mathbf{PQ} - \mathbf{QP} = \mathbf{P}$, then $\mathbf{PQ} = \mathbf{QP} + \mathbf{P}$, and so $\mathbf{P}^2\mathbf{Q} = \mathbf{P}(\mathbf{PQ}) = \mathbf{P}(\mathbf{QP} + \mathbf{P}) = (\mathbf{PQ})\mathbf{P} + \mathbf{P}^2 = (\mathbf{QP}+\mathbf{P})\mathbf{P}+\mathbf{P}^2 = \mathbf{QP}^2+2\mathbf{P}^2$. Thus, $\mathbf{P}^2\mathbf{Q}-\mathbf{QP}^2 = 2\mathbf{P}^2$. Moreover, $\mathbf{P}^3\mathbf{Q} = \mathbf{P}(\mathbf{P}^2\mathbf{Q}) = \mathbf{P}(\mathbf{QP}^2+2\mathbf{P}^2) = (\mathbf{PQ})\mathbf{P}^2 + 2\mathbf{P}^3 = (\mathbf{QP} + \mathbf{P})\mathbf{P}^2 + 2\mathbf{P}^3 = \mathbf{QP}^3 + 3\mathbf{P}^3$ Hence, $\mathbf{P}^3\mathbf{Q} - \mathbf{QP}^3 = 3\mathbf{P}^3$.

16 Determinants and Inverse Matrices

16.1

3. (a) Cramer's rule gives $x = \dfrac{\begin{vmatrix} 8 & -1 \\ 5 & -2 \end{vmatrix}}{\begin{vmatrix} 3 & -1 \\ 1 & -2 \end{vmatrix}} = \dfrac{-16+5}{-6+1} = \dfrac{11}{5}$, $\quad y = \dfrac{\begin{vmatrix} 3 & 8 \\ 1 & 5 \end{vmatrix}}{\begin{vmatrix} 3 & -1 \\ 1 & -2 \end{vmatrix}} = \dfrac{15-8}{-5} = -\dfrac{7}{5}$.

(b) and (c) are done in the same way.

7. (b) Note that because c_1 is the proportion of income consumed, we can assume that $0 \le c_1 \le 1$. Likewise, $0 \le c_2 \le 1$. Because $m_1 \ge 0$ and $m_2 \ge 0$, we see that $D > 0$ (excluding the case $c_1 = c_2 = 1$).
(c) Y_2 depends linearly on A_1. Increasing A_1 by one unit changes Y_2 by the factor $m_1/D \ge 0$, so Y_2 increases when A_1 increases.

Here is an economic explanation: An increase in A_1 increases nation 1's income, Y_1. This in turn increases nation 1's imports, M_1. However, nation 1's imports are nation 2's exports, so this causes nation 2's income, Y_2, to increase, and so on.

16.2

1. (a) Sarrus's rule yields: $\begin{vmatrix} 1 & -1 & 0 \\ 1 & 3 & 2 \\ 1 & 0 & 0 \end{vmatrix} = 0 - 2 + 0 - 0 - 0 - 0 = -2.$

(b) $\begin{vmatrix} 1 & -1 & 0 \\ 1 & 3 & 2 \\ 1 & 2 & 1 \end{vmatrix} = 3 - 2 - 0 - 0 - 4 - (-1) = -2$

(c) 5 of the 6 products have 0 as a factor. The only product that does not include 0 as a factor is the product of the terms on the main diagonal. The determinant is therefore adf.

(d) $\begin{vmatrix} a & 0 & b \\ 0 & e & 0 \\ c & 0 & d \end{vmatrix} = aed + 0 + 0 - bec - 0 - 0 = e(ad - bc)$

3. (a) The determinant of the coefficient matrix is $|\mathbf{A}| = \begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & -1 & -1 \end{vmatrix} = -4.$

The numerators in (16.2.4) are (verify!)

$$\begin{vmatrix} 2 & -1 & 1 \\ 0 & 1 & -1 \\ -6 & -1 & -1 \end{vmatrix} = -4, \quad \begin{vmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ -1 & -6 & -1 \end{vmatrix} = -8, \quad \begin{vmatrix} 1 & -1 & 2 \\ 1 & 1 & 0 \\ -1 & -1 & -6 \end{vmatrix} = -12$$

Hence, (4) yields the solution $x_1 = 1$, $x_2 = 2$, and $x_3 = 3$. Inserting this into the original system of equations confirms that this is a correct answer.
(b) The determinant of the coefficient matrix is equal to -2, and the numerators in (16.2.4) are all 0, so the unique solution is $x_1 = x_2 = x_3 = 0$. (c). Follow the pattern in (a).

6. (a) Substituting $T = d + tY$ into the expression for C gives $C = a - bd + b(1 - t)Y$. Substituting for C in the expression for Y then yields $Y = a + b(Y - d - tY) + A_0$. Then solve for Y, T, and C in turn to derive the answers given in (b) below.

(b) We write the system as $\begin{pmatrix} 1 & -1 & 0 \\ -b & 1 & b \\ -t & 0 & 1 \end{pmatrix} \begin{pmatrix} Y \\ C \\ T \end{pmatrix} = \begin{pmatrix} A_0 \\ a \\ d \end{pmatrix}$. Then Cramer's rule yields

$$Y = \frac{\begin{vmatrix} A_0 & -1 & 0 \\ a & 1 & b \\ d & 0 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 0 \\ -b & 1 & b \\ -t & 0 & 1 \end{vmatrix}} = \frac{a - bd + A_0}{1 - b(1 - t)}, \quad C = \frac{\begin{vmatrix} 1 & A_0 & 0 \\ -b & a & b \\ -t & d & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 0 \\ -b & 1 & b \\ -t & 0 & 1 \end{vmatrix}} = \frac{a - bd + A_0 b(1 - t)}{1 - b(1 - t)}$$

$$T = \frac{\begin{vmatrix} 1 & -1 & A_0 \\ -b & 1 & a \\ -t & 0 & d \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 0 \\ -b & 1 & b \\ -t & 0 & 1 \end{vmatrix}} = \frac{t(a + A_0) + (1 - b)d}{1 - b(1 - t)}$$

(This problem is meant to train you in using Cramer's rule. Note how systematic elimination is much more efficient.)

16.3

1. Each of the determinants is a sum of $4! = 24$ terms. In (a) there is only one nonzero term. In fact, according to (16.3.4), the value of the determinant is $abcd$. (b) Only two terms in the sum are nonzero: The product of the elements on the main diagonal, which is $1 \cdot 1 \cdot 1 \cdot d$, with a plus sign, and the term shown here:

$$\begin{vmatrix} 1 & 0 & 0 & \boxed{1} \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ \boxed{a} & b & c & d \end{vmatrix}$$

Since there are 5 rising lines between the pairs, the sign of the product $1 \cdot 1 \cdot 1 \cdot a$ must be minus. So the value of the determinant is $d - a$. (c) 4 terms are nonzero. See the text.

16.4

10. (a) and (b) see the text. (c) We have $(\mathbf{I}_n - \mathbf{A})(\mathbf{I}_n + \mathbf{A}) = \mathbf{I}_n \cdot \mathbf{I}_n - \mathbf{A}\mathbf{I}_n + \mathbf{I}_n\mathbf{A} - \mathbf{A}\mathbf{A} = \mathbf{I}_n - \mathbf{A} + \mathbf{A} - \mathbf{A}^2 = \mathbf{I}_n - \mathbf{A}^2$, and this expression equals $\mathbf{0}$ if and only if $\mathbf{A}^2 = \mathbf{I}_n$.

12. The description in the answer in the text amounts to the following:

$$
D_n = \begin{vmatrix} a+b & a & \cdots & a \\ a & a+b & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & a \end{vmatrix}
\begin{matrix} \leftarrow & \cdots & \leftarrow \\ 1 & \cdots & \\ & \ddots & \\ & & 1 \end{matrix}
= \begin{vmatrix} na+b & na+b & \cdots & na+b \\ a & a+b & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & a+b \end{vmatrix}
$$

$$
= (na+b)\begin{vmatrix} 1 & 1 & \cdots & 1 \\ a & a+b & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & a+b \end{vmatrix}
\begin{matrix} -a & \cdots & -a \\ \leftarrow & \cdots & \\ & \ddots & \\ \leftarrow & & \end{matrix}
= (na+b)\begin{vmatrix} 1 & 1 & \cdots & 1 \\ 0 & b & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b \end{vmatrix}
$$

According to (16.3.4), the last determinant is b^{n-1}. Thus $D_n = (na+b)b^{n-1}$.

16.5

1. (a) See the answer. (b) One possibility is to expand by the second row or the third column (because they have both two zero entries). But it is easier first to use elementary operations to get a row or a column with one more zero. For instance in this way:

$$
\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & 0 & 11 \\ 2 & -1 & 0 & 3 \\ -2 & 0 & -1 & 3 \end{vmatrix}
\begin{matrix} -2 & 2 \\ \\ \leftarrow & \\ \leftarrow & \end{matrix}
= \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & 0 & 11 \\ 0 & -5 & -6 & -5 \\ 0 & 4 & 5 & 11 \end{vmatrix}
$$

$$
= \begin{vmatrix} -1 & 0 & 11 \\ -5 & -6 & -5 \\ 4 & 5 & 11 \end{vmatrix}
\begin{matrix} -5 & -4 \\ \leftarrow & \\ \leftarrow & \end{matrix}
= \begin{vmatrix} -1 & 0 & 11 \\ 0 & -6 & -60 \\ 0 & 5 & 55 \end{vmatrix}
$$

$$
= -1 \begin{vmatrix} -6 & -60 \\ 5 & 55 \end{vmatrix} = -(-330+300) = 30
$$

When computing determinants one can use elementary row operations as well as column operations, but column operations become meaningless when solving linear equation systems using Gaussian elimination. When elementary operations have produced a row or column with only one non-zero element, then it is natural to expand the determinant by that row or column. (c) See the answer in the text.

16.6

6. (b) From (a), $\mathbf{A}^3 - 2\mathbf{A}^2 + \mathbf{A} = \mathbf{I}$, or $\mathbf{A}(\mathbf{A}^2 - 2\mathbf{A} + \mathbf{I}) = \mathbf{I}$, so using Theorem 16.6.2, $\mathbf{A}^{-1} = \mathbf{A}^2 - 2\mathbf{A} + \mathbf{I}$. The last expression can also be written $(\mathbf{A} - \mathbf{I})^2$. (c) See the text.

9. $\mathbf{B}^2 + \mathbf{B} = \begin{pmatrix} 3/2 & -5 \\ -1/4 & 3/2 \end{pmatrix} + \begin{pmatrix} -1/2 & 5 \\ 1/4 & -1/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$. One can verify directly that $\mathbf{B}^3 - 2\mathbf{B} + \mathbf{I} = \mathbf{0}$, but here is an alternative that makes use of $\mathbf{B}^2 + \mathbf{B} = \mathbf{I}$: $\mathbf{B}^3 - 2\mathbf{B} + \mathbf{I} = \mathbf{B}^3 + \mathbf{B}^2 - \mathbf{B}^2 - 2\mathbf{B} + \mathbf{I} = \mathbf{B}(\mathbf{B}^2 + \mathbf{B}) - \mathbf{B}^2 - 2\mathbf{B} + \mathbf{I} = \mathbf{B} - \mathbf{B}^2 - 2\mathbf{B} + \mathbf{I} = -(\mathbf{B}^2 + \mathbf{B}) + \mathbf{I} = \mathbf{0}$.

16.7

1. (a) The determinant is $10-12=-2$, and the adjoint is $\begin{pmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{pmatrix} = \begin{pmatrix} 5 & -3 \\ -4 & 2 \end{pmatrix}$, so the inverse is

$$-\tfrac{1}{2}\begin{pmatrix} 5 & -3 \\ -4 & 2 \end{pmatrix} = \begin{pmatrix} -5/2 & 3/2 \\ 2 & -1 \end{pmatrix}$$

(b) If we denote the matrix by $\mathbf{A}$, the adjoint is

$$\operatorname{adj}\mathbf{A} = \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix} = \begin{pmatrix} 1 & 4 & 2 \\ 2 & -1 & 4 \\ 4 & -2 & -1 \end{pmatrix}$$

and the determinant is $|\mathbf{A}| = a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} = 1\cdot 1 + 2\cdot 4 + 0\cdot 2 = 9$, (by expansion along the first column). Hence,

$$\mathbf{A}^{-1} = \tfrac{1}{9}(\operatorname{adj}\mathbf{A}) = \tfrac{1}{9}\begin{pmatrix} 1 & 4 & 2 \\ 2 & -1 & 4 \\ 4 & -2 & -1 \end{pmatrix}$$

(c) Since the determinant is 0 there is no inverse.

3. The determinant of $\mathbf{I}-\mathbf{A}$ is $|\mathbf{I}-\mathbf{A}| = 0.496$, and the adjoint is $\operatorname{adj}(\mathbf{I}-\mathbf{A}) = \begin{pmatrix} 0.72 & 0.64 & 0.40 \\ 0.08 & 0.76 & 0.32 \\ 0.16 & 0.28 & 0.64 \end{pmatrix}$.

Hence $(\mathbf{I}-\mathbf{A})^{-1} = \dfrac{1}{0.496}\cdot\operatorname{adj}(\mathbf{I}-\mathbf{A}) \approx \begin{pmatrix} 1.45161 & 1.29032 & 0.80645 \\ 0.16129 & 1.53226 & 0.64516 \\ 0.32258 & 0.56452 & 1.29032 \end{pmatrix}$, rounded to five decimal places. If you want an exact answer, note that $\dfrac{1000}{496} = \dfrac{125}{62}$ and $\operatorname{adj}(\mathbf{I}-\mathbf{A}) = \begin{pmatrix} 0.72 & 0.64 & 0.40 \\ 0.08 & 0.76 & 0.32 \\ 0.16 & 0.28 & 0.64 \end{pmatrix} = \dfrac{1}{25}\begin{pmatrix} 18 & 16 & 10 \\ 2 & 19 & 8 \\ 4 & 7 & 16 \end{pmatrix}$. This gives $(\mathbf{I}-\mathbf{A})^{-1} = \dfrac{5}{62}\begin{pmatrix} 18 & 16 & 10 \\ 2 & 19 & 8 \\ 4 & 7 & 16 \end{pmatrix}$.

4. Let $\mathbf{B}$ denote the $n\times p$ matrix whose ith column has the elements $b_{1i}, b_{2i}, \dots, b_{ni}$. The p systems of n equations in n unknowns can be expressed as $\mathbf{AX}=\mathbf{B}$, where $\mathbf{A}$ is $n\times n$ and $\mathbf{X}$ is $n\times p$. Following the method illustrated in Example 2, exactly the same row operations that transform the $n\times 2n$ matrix $(\mathbf{A} : \mathbf{I})$ into $(\mathbf{I} : \mathbf{A}^{-1})$ will also transform the $n\times(n+p)$ matrix $(\mathbf{A} : \mathbf{B})$ into $(\mathbf{I} : \mathbf{B}^*)$, where $\mathbf{B}^*$ is the matrix with elements b^*_{ij}. (In fact, because these row operations are together equivalent to premultiplication by $\mathbf{A}^{-1}$, it must be true that $\mathbf{B}^* = \mathbf{A}^{-1}\mathbf{B}$.) When $k=r$, the solution to the system is $x_1 = b^*_{1r}, x_2 = b^*_{2r}, \dots, x_n = b^*_{nr}$.

5. (a) The following shows that the inverse is $\begin{pmatrix} -2 & -1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$:

$$\left(\begin{array}{cc:cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right) \begin{matrix} -3 \\ \leftarrow \end{matrix} \sim \left(\begin{array}{cc:cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right) \begin{matrix} \\ -\frac{1}{2} \end{matrix} \sim \left(\begin{array}{cc:cc} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right) \begin{matrix} \leftarrow \\ -2 \end{matrix} \sim \left(\begin{array}{cc:cc} 1 & 2 & -2 & -1 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right)$$

(b)

$$
\left(\begin{array}{ccc:ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 4 & 5 & 0 & 1 & 0 \\ 3 & 5 & 6 & 0 & 0 & 1 \end{array}\right) \begin{matrix} -2 \quad -3 \\ \leftarrow \\ \leftarrow \end{matrix} \sim \left(\begin{array}{ccc:ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 & 1 & 0 \\ 0 & -1 & -3 & -3 & 0 & 1 \end{array}\right) \begin{matrix} \\ \leftarrow \\ \leftarrow \end{matrix}
$$

$$
\sim \left(\begin{array}{ccc:ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -3 & -3 & 0 & 1 \\ 0 & 0 & -1 & -2 & 1 & 0 \end{array}\right) \begin{matrix} \\ -1 \\ -1 \end{matrix} \sim \left(\begin{array}{ccc:ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3 & 3 & 0 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right) \begin{matrix} \leftarrow \\ -2 \\ \\ \end{matrix}
$$

$$
\sim \left(\begin{array}{ccc:ccc} 1 & 0 & -3 & -5 & 0 & 2 \\ 0 & 1 & 3 & 3 & 0 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right) \begin{matrix} \leftarrow \\ \leftarrow \\ -3 \quad 3 \end{matrix} \sim \left(\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & -3 & 2 \\ 0 & 1 & 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right)
$$

(c) We see that the third row is the fist row multiplied by -3, so the matrix has no inverse.

16.8

1. (a) The determinant $|\mathbf{A}|$ of the coefficient matrix is $|\mathbf{A}| = \begin{vmatrix} 1 & 2 & -1 \\ 2 & -1 & 1 \\ 1 & -1 & -3 \end{vmatrix} = 19$.

The determinants in (16.8.2) are (verify!)

$$
\begin{vmatrix} -5 & 2 & -1 \\ 6 & -1 & 1 \\ -3 & -1 & -3 \end{vmatrix} = 19\,, \qquad \begin{vmatrix} 1 & -5 & -1 \\ 2 & 6 & 1 \\ 1 & -3 & -3 \end{vmatrix} = -38\,, \qquad \begin{vmatrix} 1 & 2 & -5 \\ 2 & -1 & 6 \\ 1 & -1 & -3 \end{vmatrix} = 38
$$

According to (16.8.4) the solution is $x = 19/19 = 1$, $y = -38/19 = -2$, and $z = 38/19 = 2$. Inserting this into the original system of equations confirms that this is the correct answer.

(b) The determinant $|\mathbf{A}|$ of the coefficient matrix is $\begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{vmatrix} = -1$.

The determinants in (16.8.2) are (verify!)

$$
\begin{vmatrix} 3 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 6 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{vmatrix} = 3\,, \qquad \begin{vmatrix} 1 & 3 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 6 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{vmatrix} = -6\,, \qquad \begin{vmatrix} 1 & 1 & 3 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 6 & 1 \\ 0 & 1 & 1 & 1 \end{vmatrix} = -5\,, \qquad \begin{vmatrix} 1 & 1 & 0 & 2 \\ 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 6 \\ 0 & 1 & 0 & 1 \end{vmatrix} = 5
$$

According to (16.8.4) the solution is $x = -3$, $y = 6$, $z = 5$, and $u = -5$. Inserting this into the original system of equations confirms that this is the correct answer.

3. According to Theorem 16.8.2, the system has nontrivial solutions iff the determinant of the coefficient *equal* to 0. Expansion according to the first row gives

$$
\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = a \begin{vmatrix} c & a \\ a & b \end{vmatrix} - b \begin{vmatrix} b & a \\ c & b \end{vmatrix} + c \begin{vmatrix} b & c \\ c & a \end{vmatrix}
$$

$$
= a(bc - a^2) - b(b^2 - ac) + c(ab - c^2) = 3abc - a^3 - b^3 - c^3.
$$

Thus the system has nontrivial solutions iff $\quad 3abc - a^3 - b^3 - c^3 = 0$.

16.9

4. The equation system is obtained directly from (16.9.4).

Review Problems for Chapter 16

3. It is a bad idea to use "brute force" here. Note instead that rows 1 and 3 and rows 2 and 4 in the determinant have "much in common". So begin by subtracting row 3 from row 1, and row 4 from row 2. According to Theorem 16.4.1(F), this does not change the value of the determinant. This gives, if we thereafter use Theorem 16.4.1(C),

$$\begin{vmatrix} 0 & a-b & 0 & b-a \\ b-a & 0 & a-b & 0 \\ x & b & x & a \\ a & x & b & x \end{vmatrix} = (a-b)^2 \begin{vmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ x & b & x & a \\ a & x & b & x \end{vmatrix} = (a-b)^2 \begin{vmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ x & b & x & a+b \\ a & x & b & 2x \end{vmatrix}$$

The last equality is obtained by adding row 2 to row 4 in the middle determinant. If we expand the last determinant by the row 1, we end up with an easy 3×3 determinant to evaluate. The equation becomes $(a-b)^2(4x^2 - (a+b)^2) = (a-b)^2(2x + (a+b))(2x - (a+b)) = 0$. The conclusion follows.

5. (a) Expanding by column 3: $|\mathbf{A}| = \begin{vmatrix} q & -1 & q-2 \\ 1 & -p & 2-p \\ 2 & -1 & 0 \end{vmatrix} = (q-2)\begin{vmatrix} 1 & -p \\ 2 & -1 \end{vmatrix} - (2-p)\begin{vmatrix} q & -1 \\ 2 & -1 \end{vmatrix} =$ $(q-2)(-1+2p) - (2-p)(-q+2) = (q-2)(p+1)$, but there are many other ways.
$|\mathbf{A}+\mathbf{E}| = \begin{vmatrix} q+1 & 0 & q-1 \\ 2 & 1-p & 3-p \\ 3 & 0 & 1 \end{vmatrix} = (1-p)\begin{vmatrix} q+1 & q-1 \\ 3 & 1 \end{vmatrix} = (1-p)[q+1-3(q-1)]$ $= 2(p-1)(q-2)$. For the rest see the text.

8. (a) Note that

$$\mathbf{U}^2 = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix} = \begin{pmatrix} n & n & \dots & n \\ n & n & \dots & n \\ \vdots & \vdots & \ddots & \vdots \\ n & n & \dots & n \end{pmatrix} = n\mathbf{U}$$

(b) The trick is to note that

$$\mathbf{A} = \begin{pmatrix} 4 & 3 & 3 \\ 3 & 4 & 3 \\ 3 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix} = \mathbf{I}_3 + 3\mathbf{U}$$

From (a), $(\mathbf{I}_3 + 3\mathbf{U})(\mathbf{I}_3 + b\mathbf{U}) = \mathbf{I}_3 + (3 + b + 3 \cdot 3b\mathbf{U}) = \mathbf{I}_3 + (3 + 10b)\mathbf{U}$, which is equal to $\mathbf{I}_3$ if we choose $b = -3/10$. It follows that

$$\mathbf{A}^{-1} = (\mathbf{I}_3 + 3\mathbf{U})^{-1} = \mathbf{I}_3 - (3/10)\mathbf{U} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{3}{10} & \frac{3}{10} & \frac{3}{10} \\ \frac{3}{10} & \frac{3}{10} & \frac{3}{10} \\ \frac{3}{10} & \frac{3}{10} & \frac{3}{10} \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 7 & -3 & -3 \\ -3 & 7 & -3 \\ -3 & -3 & 7 \end{pmatrix}$$

11. (a) Gauss-elimination yields

$$\begin{pmatrix} a & 1 & 4 & 2 \\ 2 & 1 & a^2 & 2 \\ 1 & 0 & -3 & a \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -3 & a \\ 2 & 1 & a^2 & 2 \\ a & 1 & 4 & 2 \end{pmatrix} \begin{matrix} -2 & -a \\ & \\ & \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 0 & -3 & a \\ 0 & 1 & a^2+6 & -2a+2 \\ 0 & 1 & 3a+4 & -a^2+2 \end{pmatrix} \begin{matrix} \\ -1 \\ \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 0 & -3 & a \\ 0 & 1 & a^2+6 & -2a+2 \\ 0 & 0 & -a^2+3a-2 & -a^2+2a \end{pmatrix}$$

It follows that the system has unique solution iff $-a^2+3a-2 \neq 0$, i.e. iff $a \neq 1$ and $a \neq 2$.

If $a = 2$, the last row consists only of 0's so there ia an infinite number of solutions, while if $a = 1$, there are no solutions.

(b) If we perform the same elementary operations as in (a) on the associated extended matrix, we get

$$\begin{pmatrix} 1 & 0 & -3 & b_3 \\ 0 & 1 & a^2+6 & b_2 - 2b_3 \\ 0 & 0 & -a^2+3a-2 & b_1 - b_2 + (2-a)b_3 \end{pmatrix},$$

We see that there are infinitely many solutions iff all elements in the last row are 0, i.e. iff $a = 1$ and $b_1 - b_2 + b_3 = 0$, or when $a = 2$ and $b_1 = b_2$.

13. For once we use "unsystematic elimination". Solve the first equation for y, the second for z, and the fourth for u, using the expression found for y. Insert all this into the third equation. This gives: $a(b-2)x = -2a+2b+3$. There is a unique solution provided $a(b-2) \neq 1$. We easily verify the solutions in the text.

15. We prove the result for 3×3 matrices that differ only in the first row:

$$\begin{vmatrix} a_{11}+x & a_{12}+y & a_{13}+z \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} x & y & z \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \qquad (*)$$

We expand the first determinant in $(*)$ by the first row, where C_{11}, C_{12}, C_{13} are the complements of the three entries in the first row, and we get

$$(a_{11}+x)C_{11} + (a_{12}+y)C_{12} + (a_{13}+z)C_{13} = [a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}] + [xC_{11} + yC_{12} + zC_{13}]$$

The sums in square brackets are the two last determinants in $(*)$.

17 Linear Programming

17.1

3. The set A is the shaded set in Fig. SM17.1.3.

(a) The solution is obviously at the point P in the figure because it has the largest x_2 coordinate among all points in A. P is the point where the two lines $-2x_1 + x_2 = 2$ and $x_1 + 2x_2 = 8$ intersect, and the solution of these two equations is $(x_1, x_2) = (4/5, 18/5)$.

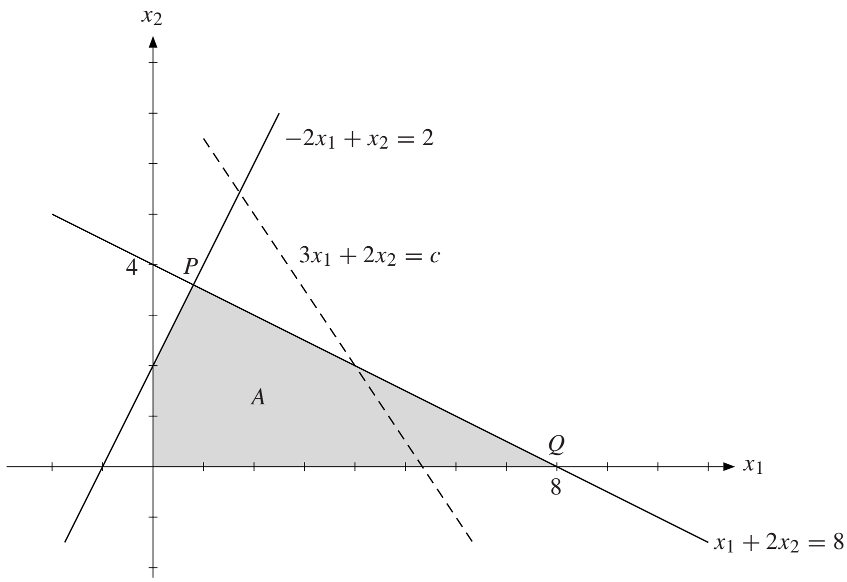


Figure SM17.1.3

(b) The point in A with the largest x_1 coordinate is obviously $Q = (8, 0)$.
(c) One of the lines $3x_1 + 2x_2 = c$ is the dashed line in Fig. SM17.1.3. As c increases, the line moves out farther and farther to the north-east. The line that has the largest value of c and still has a point in common with A, is the point Q in the figure.
(d) The line $2x_1 - 2x_2 = c$ (or $x_2 = x_1 - c/2$) makes a 45° angle with the x_1 axis, and intersects the x_1 axis at $c/2$. As c decreases, the line moves to the left. The line that has the smallest value of c and still has a point in common with A, is the point P in the figure.
(e) The line $2x_1 + 4x_2 = c$ is parallel to the line $x_1 + 2x_2 = 8$ in the figure. As c increases, the line moves out farther and farther to the north-east. The line with points in common with A that has the largest value of c is obviously obtained when the line "covers" the line $x_1 + 2x_2 = 8$. So all points on the line segment between P and Q are solutions.
(f) The line $-3x_1 - 2x_2 = c$ is parallel to the dashed line in the figure, and intersects the x_1 axis at $-c/3$. As c decreases, the line moves out farther and farther to the north-east, so the solution is at $Q = (8, 0)$. (We could also argue like this: Minimizing $-3x_1 - 2x_2$ subject to $(x_1, x_2) \in A$ must occur at the same point as maximizing $3x_1 + 2x_2$ subject to $(x_1, x_2) \in A$.)

17.2

1. (a) See Fig. A17.1.1(a) in the text. When $3x_1 + 2x_2 \leq 6$ is replaced by $3x_1 + 2x_2 \leq 7$ in Problem 17.1.1, the feasible set expands because the steepest line is moved to the right. The new optimal point is at the intersection of the lines $3x_1 + 2x_2 = 7$ and $x_1 + 4x_2 = 4$, and it follows that the solution is $(x_1, x_2) = (2, 1/2)$. The old maximum value of the objective function was 36/5. The new optimal value

is $3 \cdot 2 + 4 \cdot \frac{1}{2} = 8 = 40/5$, and the difference in optimal value is $u_1^* = 5/4$.
(b) When $x_1 + 4x_2 \leq 4$ is replaced by $x_1 + 4x_2 \leq 5$, the feasible set expands because the line $x_1 + 4x_2 = 4$ is moved up. The new optimal point is at the intersection of the lines $3x_1 + 2x_2 = 6$ and $x_1 + 4x_2 = 5$, and it follows that the solution is $(x_1, x_2) = (7/5, 9/10)$. The old maximum value of the objective function was 36/5. The new optimal value is $39/5$, and the difference in optimal value is $u_2^* = 3/5$.
(c) See the text.

17.3

1. (a) From Fig. A17.3.1(a) in the text it is clear as c increases, the dashed line moves out farther and farther to the northeast. The line that has the largest value of c and still has a point in common with the feasible set is the point P, which has coordinates $(x, y) = (0, 3)$.
(b) In Fig. A17.3.1(b), as c decreases, the dashed line moves farther and farther to the south west. The line that has the smallest value of c and still has a point in common with the feasible set is the point P, which has coordinates $(u_1, u_2) = (0, 1)$. The associated minimum value is $20u_1 + 21u_2 = 21$. This is the maximum value in the primal problem, so the answer to question (c) is yes.

2. Actually not much to add. From the easily produced figure we can read off the solution.

3. See Fig. SM17.3.3 and the text.

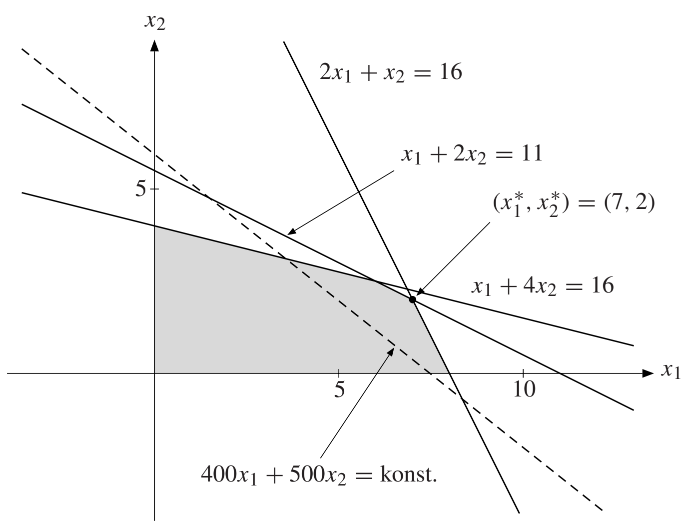


Figure SM17.3.3

17.4

2. (a) The problem is similar to Problem 17.3.3. See the answer in the text. Note that $300x_1^* + 200x_2^* = 2800$.
(b) The dual problem is

$$\min\ (54u_1 + 48u_2 + 50u_3) \quad \text{subject to} \quad \begin{cases} 6u_1 + 4u_2 + 5u_3 \geq 300 \\ 3u_1 + 6u_2 + 5u_3 \geq 200 \\ \qquad u_1, u_2, u_3 \geq 0 \end{cases}$$

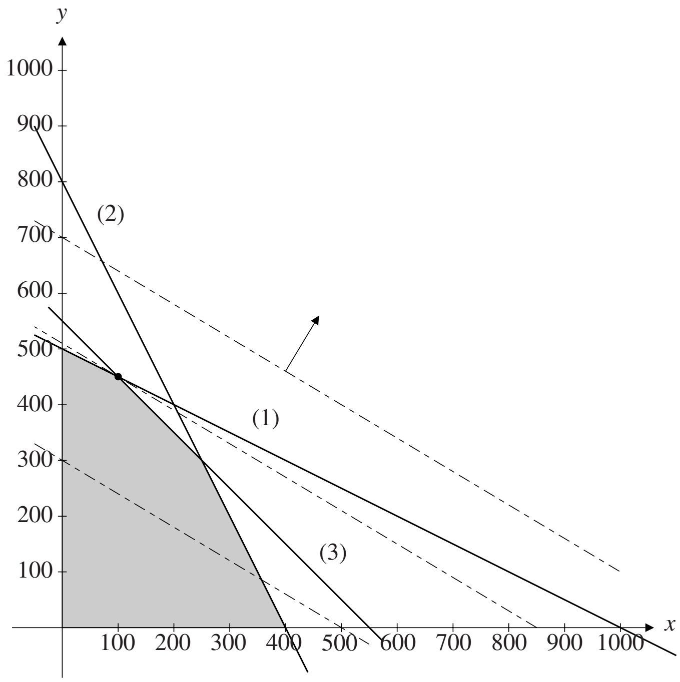


Figure SM17.5.3

The optimal solution of the primal is $x_1^* = 8$, $x_2^* = 2$. Since they are both positive, the first two constraints in the dual is satisfied with equality at the optimal triple (u_1^*, u_2^*, u_3^*). Since the second constraint in the primal is satisfied with strict inequality: $4x_1^* + 6x_2^* = 44 < 48$, $u_2^* = 0$. So $6u_1^* + 5u_3^* = 300$ and $3u_1^* + 5u_3^* = 200$. It follow that $u_1^* = 100/3$, $u_2^* = 0$, and $u_3^* = 20$, with $54u_1^* + 48u_2^* + 50u_3^* = 2800$.
(c) See the text.

17.5

3. (a) See the text. (b) The dual is given in the text. Look at Fig. SM17.5.3. We see from the figure that optimum occurs at the point where the first and the third constraint are satisfied with equality i.e. where $10x_1^* + 20x_2^* = 10\,000$ and $20x_1^* + 20x_2^* = 11\,000$. The solution is $x_1^* = 100$ and $x_2^* = 450$. The maximum value of the criterion is $300 \cdot 100 + 500 \cdot 450 = 255\,000$.
By complementary slackness, the constraints in the dual problem must in optimum be satisfied with equality. Since the second constraint in the primal in the optimum has a slack ($20 \cdot 100 + 10 \cdot 450 < 8000$), then $y_2^* = 0$. Hence $10y_1^* + 20y_3^* = 300$, $20y_1^* + 20y_3^* = 500$. It follows that the solution of the dual problem is $y_1^* = 20$, $y_2^* = 0$, $y_3^* = 5$. The minimum value of the objective function is $10\,000 \cdot 20 + 8\,000 \cdot 0 + 11\,000 \cdot 5 = 255\,000$.
(c) If the cost per hour in factory 1 increases by 100, the maximum in the primal problem would increase by $y_1^* = 20$. (The numbers y_1, y_2, and y_3 are the shadow prices for the resources in the primal.) Increasing the costs in factory 1 by 100 will therefore increase the maximum in the primal by $100 \times 20 = 2000$. (We assume that the optimal point in the primal does not change.) Since the maximum in the primal is the minimum in the dual, it follows that the minimum costs in the dual will increase by 2000 if the cost per hour in factory 1 increases by 100.

Review Problems for Chapter 17

2. (a) Regard the given LP problem as the primal and denote it by (P). Its dual is shown in answer section and is denoted by (D). If you draw the feasible set for (D) and a line $-x_1 + x_2 = c$, you see that as c increases, the line moves to the northwest, and the line that has the largest value of c and still has a point in common with the feasible set is the point (0, 8), which is the solution to (D).
(b) We see that when $x_1 = 0$ and $x_2 = 8$, the second and fourth constraint in (D) are satisfied with strict inequality, so in the optimum in (P), $y_2 = y_4 = 0$. Also, since $x_2 = 8 > 0$, the second constraint in (P) is in optimum satisfied with equality, i.e. $2y_1 - y_3 = 1$. But then we see that the objective function in (P) is $16y_1 + 6y_2 - 8y_3 - 15y_4 = 16y_1 - 8y_3 = 8(2y_1 - y_3) = 8$. If we put $y_3 = b$, $y_1 = \frac{1}{2}(1+b)$, and we conclude that $(y_1, y_2, y_3, y_4) = (\frac{1}{2}(1+b), 0, b, 0)$ must be a solution of (P) provided b is chosen such that $y_1 = \frac{1}{2}(1+b) \geq 0$, i.e. $b \geq -1$, and $y_3 = b \geq 0$, and the first constraints in (P) is satisfied. (The second constraint we already know is satisfied with equality.) The first constraint reduces to $-\frac{1}{2}(1+b) - 2b \geq -1$, or $b \leq \frac{1}{5}$. We conclude that $(\frac{1}{2}(1+b), 0, b, 0)$ is optimal provided $0 \leq b \leq \frac{1}{5}$.
(c) The objective function in (D) is now $kx_1 + x_2$. If $k \geq 0$, there is no solution. The condition for (0, 8) to be the solution is that k is less or equal to the slope of the constraint $-x_1 + 2x_2 = 16$, i.e. $k \leq -1/2$.

4. (a) See the text. (b) With the Lagrangian $\mathcal{L} = (500 - ax_1)x_1 + 250x_2 - \lambda_1(0.04x_1 + 0.03x_2 - 100) - \lambda_2(0.025x_1 + 0.05x_2 - 100) - \lambda_3(0.05x_1 - 100) - \lambda_4(0.08x_2 - 100)$, the Kuhn–Tucker conditions (with nonnegativity constraints) are: there exist numbers λ_1, λ_2, λ_3, and λ_4, such that

$$\partial\mathcal{L}/\partial x_1 = 500 - 2ax_1 - 0.04\lambda_1 - 0.025\lambda_2 - 0.05\lambda_3 \leq 0 \ (= 0 \text{ if } x_1 > 0) \quad \text{(i)}$$

$$\partial\mathcal{L}/\partial x_2 = 250 - 0.03\lambda_1 - 0.05\lambda_2 - 0.08\lambda_4 \leq 0 \ (= 0 \text{ if } x_2 > 0) \quad \text{(ii)}$$

$$\lambda_1 \geq 0, \quad \text{and} \quad \lambda_1 = 0 \quad \text{if} \quad 0.04x_1 + 0.03x_2 < 100 \quad \text{(iii)}$$

$$\lambda_2 \geq 0, \quad \text{and} \quad \lambda_2 = 0 \quad \text{if} \quad 0.025x_1 + 0.05x_2 < 100 \quad \text{(iv)}$$

$$\lambda_3 \geq 0, \quad \text{and} \quad \lambda_3 = 0 \quad \text{if} \quad 0.05x_1 < 100 \quad \text{(v)}$$

$$\lambda_4 \geq 0, \quad \text{and} \quad \lambda_4 = 0 \quad \text{if} \quad 0.08x_2 < 100 \quad \text{(vi)}$$

(c) The Kuhn–Tucker conditions are sufficient for optimality since the Lagrangian is easily seen to be concave in (x_1, x_2) for $a \geq 0$. If $(x_1, x_2) = (2000, 2000/3)$ is optimal, then (i) and (ii) are satisfied with equality. Moreover, the inequalities in (iv) and (vi) are strict when $x_1 = 2000$ and $x_2 = 2000/3$, so $\lambda_2 = \lambda_4 = 0$. Then (ii) gives $\lambda_1 = 25000/3$. It remains to check for which values of a that $\lambda_3 \geq 0$. From (i), $0.05\lambda_3 = 500 - 4000a - 0.04(25000/3) = 500/3 - 4000a \geq 0$ iff $a \leq 1/24$.